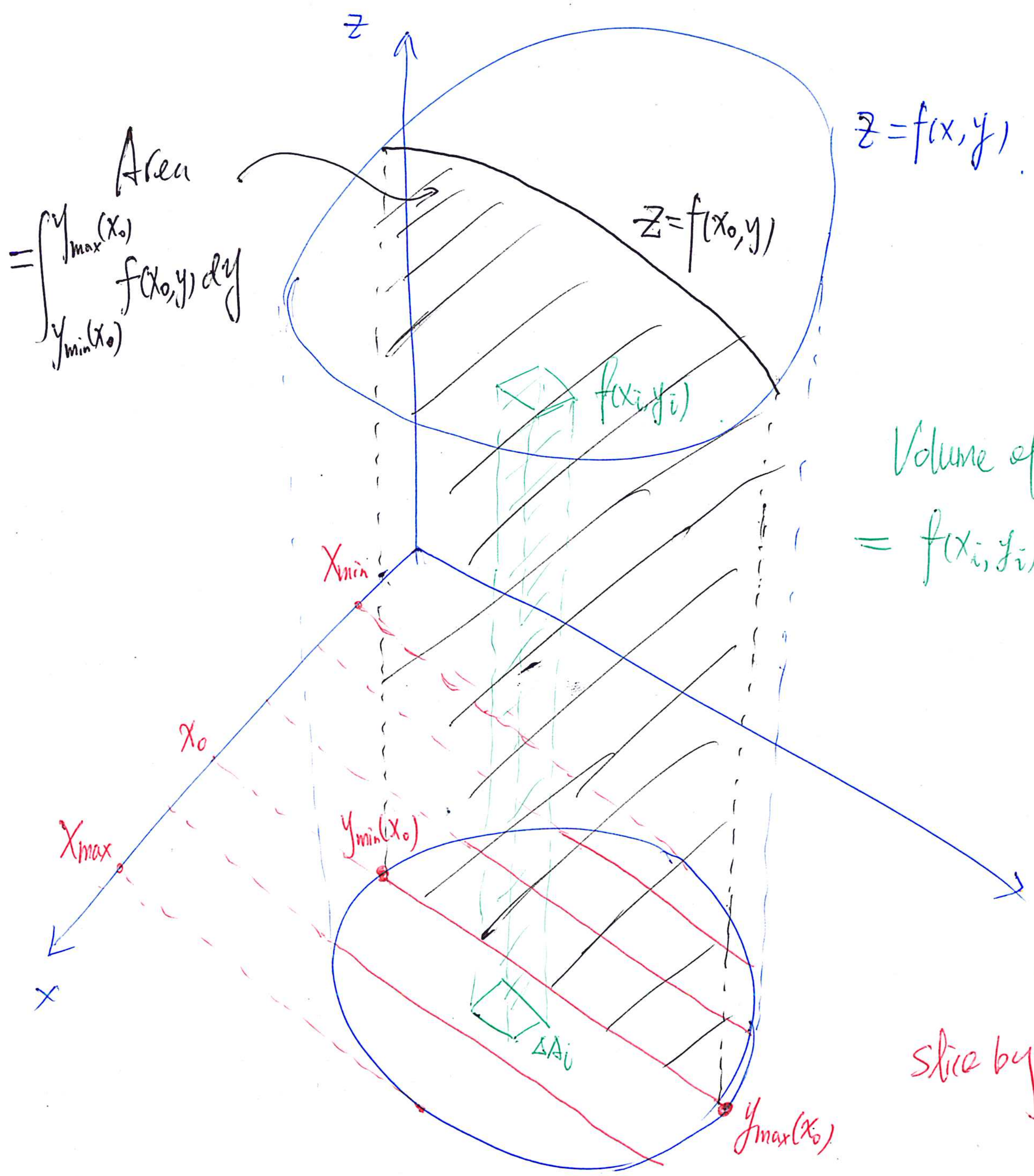
Double integral



Volume of little column

$= f(x_i,y_i)\cdot\Delta A_i$.

Slice by $x=$ constant

$$\iint_R f\,dA = \text{Volume below graph } z=f(x,y). \quad \text{"Geometrically"}$$

$$= \text{"}\lim_{\Delta A_i\to 0}\sum_i f(x_i,y_i)\,\Delta A_i\text{"} \quad \text{"Analytically"}$$

$$= \int_{x_{min}}^{x_{max}} \left[\int_{y_{min}(x)}^{y_{max}(x)} f(x,y)\,dy\right] dx \quad \text{"Computationally"}$$

← iterated integral

Computational Technique

- Exchanging order of integration.

$$\int_{x_{min}}^{x_{max}}\int_{y_{min}(x)}^{y_{max}(x)} f(x,y)\,dy\,dx = \int_{\cdot}^{\cdot}\int_{\cdot}^{\cdot} f(x,y)\,dx\,dy$$

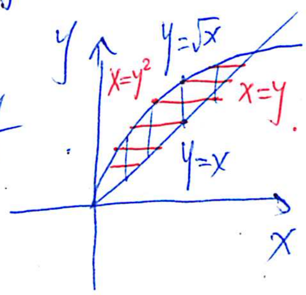
Draw picture to find the bounds.

Example. $\int_0^1\int_x^{\sqrt{x}} \frac{e^y}{y}\,dy\,dx$

$$= \int_0^1\int_{y^2}^{y} \frac{e^y}{y}\,dx\,dy$$

$$= \int_0^1 e^y - ye^y\,dy$$

$$= e-2.$$



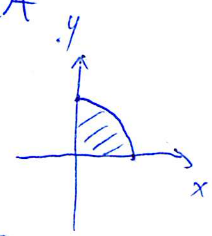
- Polar Coordinate.

$$\iint_R f\,dA = \int_{\theta_{min}}^{\theta_{max}}\int_{r_{min}(\theta)}^{r_{max}(\theta)} f(r,\theta)\,\underline{\underline{r}}\,dr\,d\theta$$

Don't forget!

Example. $\iint (1-x^2-y^2)\,dA$

R = quarter disk $\{x^2+y^2\le 1,\ x\ge 0,\ y\ge 0\}$



$$= \int_0^{\frac{\pi}{2}}\int_0^1 (1-r^2)\,r\,dr\,d\theta.$$

$$= \frac{\pi}{8}.$$

- General change of variable.

$u=u(x,y)$, $v=v(x,y)$.

Let $J = \frac{\partial(u,v)}{\partial(x,y)} = \begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix}$ "Jacobian"

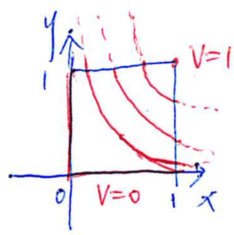
$$\boxed{\iint f(u,v)\,du\,dv = \iint f(x,y)\,|J|\,dx\,dy}$$

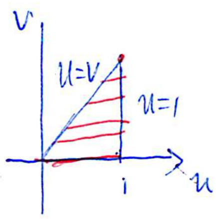
Draw pictures to find bounds.

absolute value of determinant of J.

Example. $u=x$, $v=xy$

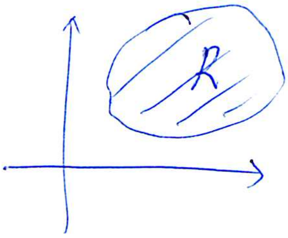
$$\int_0^1\int_0^1 x^2y\,dx\,dy = \int_0^1\int_v^1 v\,du\,dv$$



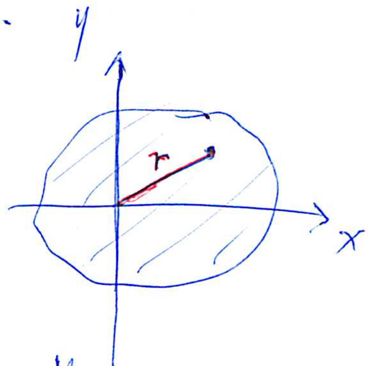


$$J = \frac{\partial(u,v)}{\partial(x,y)} = \begin{bmatrix} 1 & 0 \\ y & x \end{bmatrix}$$

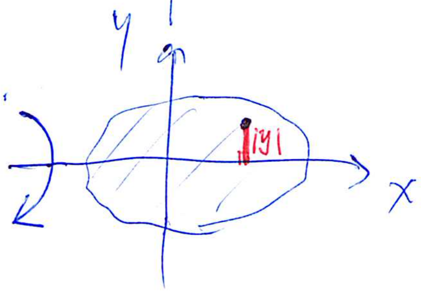
Applications



- $\text{Area}(R) = \iint_R 1\, dA$.
- $\text{Mass}(R) = \iint_R \delta\, dA$.
  (with density δ)
- Average of $f = \bar{f} = \frac{1}{\text{Area}(R)} \iint_R f\, dA$.
- Weighted average with density δ $= \frac{1}{\text{Mass}(R)} \iint_R f\delta\, dA$.
- Center of Mass: $\bar{x} = \frac{1}{\text{mass}} \iint_R x\, dA$
  $\bar{y} = \frac{1}{\text{mass}} \iint_R y\, dA$.
- Moment of Inertia about the origin $I_0 = \iint_R r^2 \delta\, dA$



Moment of Inertia about the x-axis $I_x = \iint_R y^2 \delta\, dA$.



Moment of Inertia about the y-axis $I_y = \iint_R x^2 \delta\, dA$