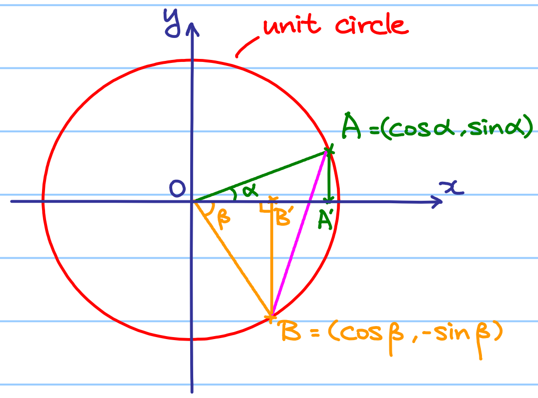
Supplementary Notes on Trigonometry



① Consider the length of AB :

1) $AB^2 = OA^2 + OB^2 - 2\cos(\alpha+\beta)$

$= 2 - 2\cos(\alpha+\beta)$

ii) $AB^2 = (AA' + BB')^2 + (A'B')^2$

$= (\sin\alpha + \sin\beta)^2 + (\cos\alpha - \cos\beta)^2$

$= 2 - 2\cos\alpha\cos\beta + 2\sin\alpha\sin\beta$

$\therefore \cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$

② Join AB , AB cuts the x-axis at C.

Then $C = \left(\dfrac{\sin\alpha\cos\beta - \cos\alpha\sin\beta}{\sin\alpha + \sin\beta}, 0\right)$

Consider the area of ΔOAB :

i) area of $\Delta OAB = \frac{1}{2} OA \cdot OB \cdot \sin(\alpha+\beta) = \frac{1}{2}\sin(\alpha+\beta)$

ii) area of ΔOAB = area of ΔOAC + area of ΔOBC

$= \frac{1}{2} \cdot OC \cdot AA' + \frac{1}{2} \cdot OC \cdot BB'$

$= \frac{1}{2} \cdot OC \cdot (AA' + BB')$

$= \frac{1}{2} \cdot \dfrac{\sin\alpha\cos\beta + \cos\alpha\sin\beta}{\sin\alpha + \sin\beta} \cdot (\sin\alpha + \sin\beta)$

$= \frac{1}{2} \cdot (\sin\alpha\cos\beta - \cos\alpha\sin\beta)$

$\therefore \sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$
$$\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$$
replace β by $-\beta$

put $\beta=\alpha$ $\longrightarrow$ $\cos 2\alpha = \cos^2\alpha - \sin^2\alpha$
$= 2\cos^2\alpha - 1 = 1 - 2\sin^2\alpha$

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$
$$\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$$
replace β by $-\beta$

put $\beta=\alpha$ $\longrightarrow$ $\sin 2\alpha = 2\sin\alpha\cos\alpha$

taking quotient of the 1st and the 3rd eq^n

$$\tan(\alpha+\beta) = \frac{\tan\alpha + \tan\beta}{1-\tan\alpha\tan\beta}$$

put $\beta=\alpha$ $\longrightarrow$ $\tan 2\alpha = \dfrac{2\tan\alpha}{1-\tan^2\alpha}$

$$\tan(\alpha-\beta) = \frac{\tan\alpha - \tan\beta}{1+\tan\alpha\tan\beta}$$

$$2\cos\alpha\cos\beta = \cos(\alpha+\beta) + \cos(\alpha-\beta)$$
$$-2\sin\alpha\sin\beta = \cos(\alpha+\beta) - \cos(\alpha-\beta)$$
$$2\sin\alpha\cos\beta = \sin(\alpha+\beta) + \sin(\alpha-\beta)$$
$$2\cos\alpha\sin\beta = \sin(\alpha+\beta) - \sin(\alpha-\beta)$$

put $\alpha = \frac{A+B}{2}$, $\beta = \frac{A-B}{2}$

$$\sin A + \sin B = 2\sin\frac{A+B}{2}\cos\frac{A-B}{2}$$
$$\sin A - \sin B = 2\cos\frac{A+B}{2}\sin\frac{A-B}{2}$$
$$\cos A + \cos B = 2\cos\frac{A+B}{2}\cos\frac{A-B}{2}$$
$$\cos A - \cos B = -2\sin\frac{A+B}{2}\sin\frac{A-B}{2}$$