THE CHINESE UNIVERSITY OF HONG KONG
DEPARTMENT OF MATHEMATICS

MATH1010 I/J University Mathematics 2015-2016
Suggested Solution to Problem Set 2

1. (a) $\lim_{n\to\infty} \frac{3n^2-2n+7}{2n^2+3} = \lim_{n\to\infty} \frac{3-\frac{2}{n}+\frac{7}{n^2}}{2+\frac{3}{n}} = \frac{\lim_{n\to\infty} 3-\frac{2}{n}+\frac{7}{n^2}}{\lim_{n\to\infty} 2+\frac{3}{n}} = \frac{3}{2}$

(b) $\lim_{n\to\infty} \frac{-3n^2}{\sqrt[3]{27n^6-5n+1}} = \lim_{n\to\infty} \frac{-3}{(\frac{1}{n^2})\sqrt[3]{27n^6-5n+1}} = \lim_{n\to\infty} \frac{-3}{\sqrt[3]{27-\frac{5}{n^5}+\frac{1}{n^6}}} = \frac{-3}{\sqrt[3]{27}} = -1$

(c)

$$\begin{aligned}
\lim_{n\to\infty} \sqrt{4n^2+n}-\sqrt{4n^2-1} &= \lim_{n\to\infty} (\sqrt{4n^2+n}-\sqrt{4n^2-1})\left(\frac{\sqrt{4n^2+n}+\sqrt{4n^2-1}}{\sqrt{4n^2+n}+\sqrt{4n^2-1}}\right) \\
&= \lim_{n\to\infty} \frac{n+1}{\sqrt{4n^2+n}+\sqrt{4n^2-1}} \\
&= \lim_{n\to\infty} \frac{1+\frac{1}{n}}{\sqrt{4+\frac{1}{n}}+\sqrt{4-\frac{1}{n^2}}} \\
&= \frac{1}{4}
\end{aligned}$$

(d) Note that $-2 \le \sin(2^n)+(-1)^n\cos(2^n) \le 2$ for all natural numbers n. Therefore,

$$-\frac{2}{n} \le \frac{\sin(2^n)+(-1)^n\cos(2^n)}{n^3} \le \frac{2}{n}$$

for all natural numbers n.
Also, we have $\lim_{n\to\infty} -\frac{2}{n} = \lim_{n\to\infty} \frac{2}{n} = 0$.
Therefore, by the sandwich theorem, $\lim_{n\to\infty} \frac{\sin(2^n)+(-1)^n\cos(2^n)}{n^3} = 0$.

2. (a) Let $P(n)$ be the statement that "$a_n \le 3$".

- When $n = 1$, $a_1 = 1 \le 3$. Therefore, $P(1)$ is true.
- Suppose $P(n)$ is true for some natural number n, i.e. $a_n \le 3$.
  Then,

$$\begin{aligned}
a_{n+1}-3 &= \frac{12a_n+12}{a_n+13}-3 \\
&= \frac{9a_n-27}{a_n+13} \\
&= \frac{9(a_n-3)}{a_n+13} \\
&\le 0 \qquad (\because 0 < a_n \le 3)
\end{aligned}$$

  Therefore, $P(n+1)$ is true.

By mathematical induction, $a_n \le 3$ for all natural numbers n.

(b) Let $P(n)$ be the statement that "$a_{n+1} \geq a_n$".

- When $n = 1$, $a_2 = \frac{12}{7} \geq 1 = a_1$. Therefore, $P(1)$ is true.
- Suppose $P(n)$ is true for some natural number n, i.e. $a_{n+1} \geq a_n$.
  Then,

$$\begin{aligned} a_{n+2} - a_{n+1} &= \frac{12a_{n+1}+12}{a_{n+1}+13} - \frac{12a_n+12}{a_n+13} \\ &= \frac{(12a_{n+1}+12)(a_n+13)-(12a_n+12)(a_{n+1}+13)}{(a_{n+1}+13)(a_n+13)} \\ &= \frac{144(a_{n+1}-a_n)}{(a_{n+1}+13)(a_n+13)} \\ &\geq 0 \qquad (\because a_{n+1} \geq a_n > 0) \end{aligned}$$

  Therefore, $P(n+1)$ is true.

By mathematical induction, $a_{n+1} \geq a_n$ for all natural numbers n, i.e. $\{a_n\}$ is a monotonic increasing sequence.

By the monotone convergence theorem, $\{a_n\}$ converges and we let $\lim_{n\to\infty} a_n$. Then,

$$\begin{aligned} \lim_{n\to\infty} a_n &= \lim_{n\to\infty} \frac{12a_{n-1}+12}{a_{n-1}+13} \\ A &= \frac{12A+12}{A+13} \\ A^2 + A - 12 &= 0 \\ A &= 3 \quad \text{or} \quad -4\,(\text{rejected}) \end{aligned}$$

Therefore, $\lim_{n\to\infty} a_n = 3$.

3. (a) Let $P(n)$ be the statement that "$\frac{2^n}{n!} \leq \frac{4}{n}$".

- When $n = 2$, LHS = RHS = 2. Therefore, $P(2)$ is true.
- Suppose $P(n)$ is true for some natural number $n \geq 2$, i.e. $\frac{2^n}{n!} \leq \frac{4}{n}$.
  Then,

$$\begin{aligned} \frac{2^{n+1}}{(n+1)!} &= \left(\frac{2}{n+1}\right)\left(\frac{2^n}{n!}\right) \\ &\leq \left(\frac{2}{n+1}\right)\left(\frac{4}{n}\right) \\ &= \left(\frac{4}{n+1}\right)\left(\frac{2}{n}\right) \\ &\leq \frac{4}{n+1} \qquad (\because n \geq 2) \end{aligned}$$

  Therefore, $P(n+1)$ is true.

By mathematical induction, $\frac{2^n}{n!} \leq \frac{4}{n}$ for all natural numbers $n \geq 2$.

(b) Note that for any natural numbers $n \geq 2$, $0 \leq \frac{2^n}{n!} \leq \frac{4}{n}$.

Also, $\lim_{n\to\infty} 0 = \lim_{n\to\infty} \frac{4}{n} = 0$.

By the sandwich theorem, $\lim_{n\to\infty} \frac{2^n}{n!}$.

4. By considering $\dfrac{1}{\sqrt{n^2+n}} \le \dfrac{1}{\sqrt{n^2+r}} \le \dfrac{1}{\sqrt{n^2+1}}$ for $r = 1, 2, 3, \cdots, n$, we have

$$\frac{1}{\sqrt{n^2+n}} \le \frac{1}{\sqrt{n^2+1}} \le \frac{1}{\sqrt{n^2+1}}$$
$$\frac{1}{\sqrt{n^2+n}} \le \frac{1}{\sqrt{n^2+2}} \le \frac{1}{\sqrt{n^2+1}}$$
$$\vdots$$
$$\frac{1}{\sqrt{n^2+n}} \le \frac{1}{\sqrt{n^2+n}} \le \frac{1}{\sqrt{n^2+1}}$$

Summing up all the above inequalities, we have

$$\frac{n}{\sqrt{n^2+n}} \le \frac{1}{\sqrt{n^2+1}} + \frac{1}{\sqrt{n^2+2}} + \cdots + \frac{1}{\sqrt{n^2+n}} \le \frac{n}{\sqrt{n^2+1}}.$$

Note that

$$\lim_{n\to\infty} \frac{n}{\sqrt{n^2+n}} = \lim_{n\to\infty} \frac{1}{\sqrt{1+\frac{1}{n}}} = 1$$
$$\lim_{n\to\infty} \frac{n}{\sqrt{n^2+1}} = \lim_{n\to\infty} \frac{1}{\sqrt{1+\frac{1}{n^2}}} = 1$$

Therefore, by the sandwich theorem,

$$\lim_{n\to\infty} \left(\frac{1}{\sqrt{n^2+1}} + \frac{1}{\sqrt{n^2+2}} + \cdots + \frac{1}{\sqrt{n^2+n}} \right) = 1.$$

5. (a)

$$\begin{aligned} x_{n+1} - y_{n+1} &= \frac{x_n^2 y_n + x_n y_n^2}{x_n^2 + y_n^2} - \frac{x_n^2 + y_n^2}{x_n + y_n} \\ &= \frac{(x_n^2 y_n + x_n y_n^2)(x_n + y_n) - (x_n^2 + y_n^2)^2}{(x_n + y_n)(x_n^2 + y_n^2)} \\ &= \frac{x_n^3 y_n + x_n y_n^3 - x_n^4 - y_n^4}{(x_n + y_n)(x_n^2 + y_n^2)} \\ &= \frac{-(x_n^3 - y_n^3)(x_n - y_n)}{(x_n + y_n)(x_n^2 + y_n^2)} \end{aligned}$$

(b) Let $P(n)$ be the statement that "$0 \le x_n \le y_n$".

- When $n = 1$, $0 \le 2 = x_1 \le 8 = y_1$. Therefore, $P(1)$ is true.
- Suppose $P(n)$ is true for some natural number n, i.e. $0 \le x_n \le y_n$. Then, $x_n^3 - y_n^3 \le 0$ and

$$x_{n+1} - y_{n+1} = \frac{-(x_n^3 - y_n^3)(x_n - y_n)}{(x_n + y_n)(x_n^2 + y_n^2)} \le 0.$$

  Therefore, $P(n+1)$ is true.

By mathematical induction, $0 \le x_n\ leqy_n$ for all natural numbers n.

Hence, for all natural numbers n, we have

$$x_{n+1} - x_n = \frac{x_n^2 y_n + x_n y_n^2}{x_n^2 + y_n^2} - x_n = \frac{-x_n^2(x_n - y_n)}{x_n^2 + y_n^2} \ge 0$$
$$y_{n+1} - y_n = \frac{x_n^2 + y_n^2}{x_n + y_n} - y_n = \frac{x_n(x_n - y_n)}{x_n + y_n} \le 0$$

Therefore, $\{x_n\}$ is a monotonic increasing sequence and $\{y_n\}$ is a monotonic decreasing sequence.

(c) For any natural number n, $x_n \le y_n \le y_{n-1} \le \cdots \le y_1 = 8$. Therefore, $\{x_n\}$ is bounded above by 8. By the monotone convergence theorem, $\{x_n\}$ converges.

(Caution: We cannot say that "For any natural number n, $x_n \le y_n$, so $\{x_n\}$ is bounded above by y_n" because y_n is not a fixed number.)

Similarly, for any natural number n, $y_n \ge x_n \ge x_{n-1} \ge \cdots \ge x_1 = 2$. Therefore, $\{y_n\}$ is bounded below by 2. By the monotone convergence theorem, $\{y_n\}$ converges.

Now, let $X = \lim_{n\to\infty} x_n$ and $Y = \lim_{n\to\infty} y_n$. Then,

$$\begin{aligned}
\lim_{n\to\infty} x_{n+1} &= \lim_{n\to\infty} \frac{x_n^2 y_n + x_n y_n^2}{x_n^2 + y_n^2} \\
X &= \frac{X^2Y + XY^2}{X^2 + Y^2} \\
X^3 + XY^2 &= X^2Y + XY^2 \\
X^2(X - Y) &= 0
\end{aligned}$$

so $X = Y$ or $X = 0$. However, $\{x_n\}$ is monotonic increasing and $x_1 = 2$ which implies that X cannot be 0.

Therefore, $X = Y$, i.e. $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n$.

(d) For any natural number n,

$$x_{n+1}y_{n+1} = \left(\frac{x_n^2 y_n + x_n y_n^2}{x_n^2 + y_n^2}\right)\left(\frac{x_n^2 + y_n^2}{x_n + y_n}\right) = x_n y_n \left(\frac{x_n + y_n}{x_n^2 + y_n^2}\right)\left(\frac{x_n^2 + y_n^2}{x_n + y_n}\right) = x_n y_n.$$

Therefore, $x_n y_n = x_{n-1}y_{n-1} = \cdots = x_2 y_2 = x_1 y_1 = (2)(8) = 16$ which is a constant.

Now, we have

$$\begin{aligned}
x_n y_n &= 16 \\
\lim_{n\to\infty} x_n y_n &= 16 \\
\left(\lim_{n\to\infty} x_n\right)\left(\lim_{n\to\infty} y_n\right) &= 16 \\
\left(\lim_{n\to\infty} x_n\right)^2 &= 16 \\
\left(\lim_{n\to\infty} x_n\right) &= 4 \quad \text{or} \quad -4\,\text{(rejected)}
\end{aligned}$$