Math 1010C Term 1 2015
Supplementary exercises 8

1. (Putnam 2003) Find the minimum value of

   $$|\sin x + \cos x + \tan x + \cot x + \sec x + \csc x|$$

   for real numbers x. (Hint: First write $\sin x+\cos x+\tan x+\cot x+\sec x+\csc x$ in terms of $\sin x$ and $\cos x$ only. Then observe that only $\sin x+\cos x$ and $\sin x\cos x$ appears, and the two are related by the identity

   $$2\sin x\cos x = (\sin x + \cos x)^2 - 1.$$

   Hence we can let say $t = \sin x + \cos x \in [-\sqrt{2}, \sqrt{2}]$, and rewrite $\sin x + \cos x + \tan x+\cot x+\sec x+\csc x$ in terms of t only. This becomes an easy one-variable minimization problem in t.)

2. (Putnam 2002) Show that, for all integers $n > 1$,

   $$\frac{1}{2ne} < \frac{1}{e} - \left(1 - \frac{1}{n}\right)^n < \frac{1}{ne}.$$

   (Hint: Rewrite the inequality as

   $$\frac{1}{e} - \frac{1}{ne} < \left(1 - \frac{1}{n}\right)^n < \frac{1}{e} - \frac{1}{2ne}$$

   and take the logarithm. Then one needs to show

   $$\log\left(1 - \frac{1}{n}\right) - 1 < n\log\left(1 - \frac{1}{n}\right) < \log\left(1 - \frac{1}{2n}\right) - 1.$$

   Establish this using power series expansion.)

3. (Putnam 1997) Let f be a twice-differentiable real-valued function satisfying

   $$f(x) + f''(x) = -xg(x)f'(x),$$

   where $g(x) \geq 0$ for all real x. Prove that $|f(x)|$ is bounded. (Hint: We use the same strategy that we adopted to prove that $\sin x$ is bounded. Just differentiate $[f(x)]^2 + [f'(x)]^2$.)