Tutorial 2

(You are NOT expected to have time to answer ALL questions.
Try as many as you can. Write down clearly your arguments.).

1. (a) (warm-up exercise on Mathematical Induction) Show, using Mathematical Induction, that

$$1 + 2^2 + 3^2 + \cdots + n^2 = \frac{n^2(n+1)^2}{4}$$

   (b) Recall that in the Generalized Binomial Theorem, we had the Binomial Coefficients, i.e.

$$C_k^a \stackrel{\text{def}}{=} \frac{a(a-1)(a-2)\cdots(a-k+1)}{k!},$$

   where a ia a real number and k a natural number (i.e. $0, 1, 2, 3, \cdots$).

   Show, using Mathematical Induction, that

$$C_0^{a+0} + C_1^{a+1} + \cdots + C_n^{a+n} = C_n^{a+n+1}$$

   where n is any natural number, i.e $n = 0, 1, 2, \cdots$

2. (Exercise on "ln" function). Denote the set of all positive real numbers by $(0, \infty)$. Show the following: The function

$$f: \ (0, \infty) \to (0, \infty)$$

   given by the rule $f(x) = \ln(1 + x^2)$ has range $R(f)$ equal to the set $(0, \infty)$.

   (Hint: you can use the following facts: (i) e^y takes on (= "can achieve") every value in the set $(1, \infty)$; (ii) for any $v > 0$, the square root $\sqrt{v}$ is always defined.)

3. In the preceding question, show that f is injective using the fact that

$$\exp(\ln(u)) = u \text{ for each } u > 0.$$

4. Find all real numbers b for which that $g(x) = x^3 + bx$ is an injective function.

5. Using elementary algebra, show that for any $x_1 > x_2 > 0$, the following inequality holds:

$$\exp(x_1) > \exp(x_2).$$

   (Hint:

   - Use

$$\exp(x_1 - x_2) = \frac{\exp(x_1)}{\exp(x_2)}$$

   - and the definition of $\exp(x_1 - x_2)$, i.e.

$$\exp(x_1 - x_2) = 1 + \frac{x_1 - x_2}{1!} + \frac{(x_1 - x_2)^2}{2!} + \cdots)$$

6. In the preceding question, is it still true that $\exp(x_1) > \exp(x_2)$ if $x_2 < x_1 < 0$?