Topics: L'Hôpital's Rule, Product Rule, Quotient Rule, Chain Rule

Things learned

- Extreme Value Theorem
- The 3 Mean Value Theorems
- L'Hôpital's Rule
- Statement of Taylor's Theorem

Things not yet learned, but may be useful

- (Product Rule for derivatives) $(f \cdot g)'(x) = f'(x)g(x) + f(x)g'(x)$. (This rule has not been proved during the last week.)
- (Quotient Rule) $(f/g)'(x) = \dfrac{f'(x)g(x) - g'(x)f(x)}{g(x)^2}$, where $g(x) \neq 0$.
- (Chain Rule)
$$\underbrace{\frac{df(y(x))}{dy}}_{\star}\frac{dy(x)}{dx} = \frac{df(y(x))}{dx} \quad ({}^1)$$

Assignments

1. (Proof of Product Rule) Let $f : (a,b) \to \mathbb{R}$ and $g : (a,b) \to \mathbb{R}$ be two functions, $\boxed{\text{both}}$ differentiable at $x = c$ in (a,b). This exercise will guide you to show (using First Principle) that
$$(f \cdot g)'(c) = f(c) \cdot g'(c) + g(c) \cdot f'(c) \tag{0.1}$$
   (a) Consider
$$\frac{f(c+h)g(c+h) - f(c)g(c)}{h}$$
   By inserting the term $\boxed{-f(c+h)g(c) + f(c+h)g(c)}$, show that the expression (**??**) can be rewritten as
$$\underbrace{\frac{f(c+h)g(c+h) - f(c+h)g(c)}{h}}_{I} + \underbrace{\frac{f(c+h)g(c) - f(c)g(c)}{h}}_{II}$$

[1] In the derivative labelled as $(\star)$, I wrote $y(x)$ in the expression '$df(y(x))$' to emphasize the fact that y is actually a function of the variable x. I could have just written y, if the reader understands this dependence on x !

(b) By taking limit $h \to 0$, find the limiting value of the term (II), i.e. $\lim_{h\to 0} \frac{f(c+h)g(c)-f(c)g(c)}{h}$.

(c) By taking limit $h \to 0$, find the limiting value of the term (I), i.e. $\lim_{h\to 0} \frac{f(c+h)g(c+h)-f(c+h)g(c)}{h}$.

(d) In (??), you had to use the following limit

$$\lim_{h\to 0} f(c+h) = f(c)$$

Give one reason why this limit is true.

(e) Combining (1a)-(1d), show that (??) is true.([2])

2. (Exercise on Chain Rule, Product Rule, Quotient Rule) In each of the following, let the function g be a twice differentiable function defined on the domain (a, b). Compute the derivative stated:

(a) $f'(x)$ if $f(x) \stackrel{\text{def}}{=} g(\frac{x}{1+(g(x))^2})$

(b) $k''(x)$ if $k(x) \stackrel{\text{def}}{=} e^{xg(x)}$

3. (Extreme Values) Write down an example of a non-differentiable function defined on $[0, 2]$ whose absolute minimum point is at $x = 1$ and absolute maximum point at $x = 2$. (Hint: By 'non-differentiable' I mean 'non-differentiable at <u>some</u> point(s) in the domain. Think simple, try 'piecewise straightline function'. There are more than one answer!)

4. (L'Hôpital's Rule) Compute the following limits:

(a) $\lim_{x\to 0^+} x^{\sin x}$
(Hint: rewrite the function in the form $\boxed{e^{\text{'some function of } x\text{'}}}$)

(b) $\lim_{x\to 0} \left(\cot(x) - \frac{1}{x}\right)$

[2](??) is usually known as the 'Product Rule'.