PROOF OF TAYLOR'S THEOREM

Here's some reflection on the proof(s) of Taylor's theorem. First we recall the (derivative form) of the theorem:

Theorem 1 (Taylor's theorem)**.** *Suppose $f\colon (a,b) \to \mathbb{R}$ is a function on (a,b), where $a, b \in \mathbb{R}$ with $a < b$. Assume that for some positive integer n, f is n-times differentiable on the open interval (a,b), and that $f, f', f'', \dots, f^{(n-1)}$ all extend continuously to the closed interval $[a,b]$ (the extended functions will still be called $f, f', f'', \dots, f^{(n-1)}$ respectively). Then there exists $c \in (a,b)$ such that*

$$f(b) = \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!}(b-a)^k + \frac{f^{(n)}(c)}{n!}(b-a)^n.$$

A key observation is that when $n = 1$, this reduces to the ordinary mean-value theorem. This suggests that we may modify the proof of the mean value theorem, to give a proof of Taylor's theorem.

The proof of the mean-value theorem comes in two parts: first, by subtracting a linear (i.e. degree 1) polynomial, we reduce to the case where $f(a) = f(b) = 0$. Next, the special case where $f(a) = f(b) = 0$ follows from Rolle's theorem.

In the proof of the Taylor's theorem below, we mimic this strategy.

The key is to observe the following generalization of Rolle's theorem:

Proposition 2. *Suppose $F\colon (a,b) \to \mathbb{R}$ is a function on (a,b), where $a, b \in \mathbb{R}$ with $a < b$. Assume that for some positive integer n, F is n-times differentiable on the open interval (a,b), and that $F, F', F'', \dots, F^{(n-1)}$ all extend continuously to the closed interval $[a,b]$ (the extended functions will still be called $F, F', F'', \dots, F^{(n-1)}$ respectively). If in addition*

$$F(a) = F'(a) = \cdots = F^{(n-1)}(a) = 0, \quad \textit{and} \quad F(b) = 0,$$

then there exists $c \in (a,b)$ such that

$$F^{(n)}(c) = 0.$$

Proof. The proof of this proposition follows readily from an n-fold application of Rolle's theorem: Since $F(a) = F(b) = 0$, by Rolle's theorem applied to F on $[a,b]$, there exists $c_1 \in (a,b)$ such that

$$F'(c_1) = 0.$$

Next, since $F'(a) = F'(c_1) = 0$, by Rolle's theorem applied to F on $[a, c_1]$, there exists $c_2 \in (a, c_1)$ such that

$$F''(c_2) = 0.$$

Repeat, then we get $c_1, \dots, c_n$ such that

$$a < c_n < c_{n-1} < \cdots < c_1 < b,$$

with

$$F^{(k)}(c_k) = 0 \quad \text{for } k = 1, 2, \dots, n.$$

In particular, setting $c = c_n$, we have $c \in (a, b)$, and

$$F^{(n)}(c) = 0.$$

□

First proof of Theorem 1. Now we apply the proposition to prove Theorem 1. The key is to construct a degree n polynomial, that allows us to reduce to the case in Proposition 2. The fact that such polynomial exists follows by a dimension counting argument in linear algebra. But we will need the explicit expression of the polynomial, so let's construct the polynomial explicitly.

Indeed, let f be as in Theorem 1. Let

$$P(x) = \sum_{k=0}^{n} a_k (x-a)^k.$$

(This is a convenient form of expressing a polynomial of degree k, since we will need to compute high order derivatives of this polynomial at the point a.) We will find coefficients $a_0, a_1, \dots, a_n$, such that $F(x) := f(x) - P(x)$ satisfies the conditions of Proposition 2. Indeed, for $k = 0, 1, \dots, n-1$, we have

$$F^{(k)}(a) = f^{(k)}(a) - k! a_k,$$

so in order for $F(a) = F'(a) = \cdots = F^{(n-1)}(a) = 0$, it suffices to set

$$a_k = \frac{f^{(k)}(a)}{k!} \quad \text{for } k = 0, 1, \dots, n-1.$$

It remains then to determine a_n. But this is determined by the equation $F(b) = 0$: indeed

$$F(b) = f(b) - \sum_{k=0}^{n} a_k (b-a)^k = f(b) - \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!}(b-a)^k - a_n (b-a)^n,$$

so setting $F(b) = 0$, we get

$$a_n = \frac{1}{(b-a)^n}\left(f(b) - \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!}(b-a)^k \right).$$

Now we have found a polynomial P such that $F(x) := f(x) - P(x)$ satisfies the conditions of Proposition 2. Hence there exists $c \in (a, b)$ such that $F^{(n)}(c) = 0$. But

$$F^{(n)}(c) = f^{(n)}(c) - P^{(n)}(c) = f^{(n)}(c) - n! a_n = f^{(n)}(c) - \frac{n!}{(b-a)^n} \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!}(b-a)^k.$$

Since $F^{(n)}(c) = 0$, it follows that

$$0 = f^{(n)}(c) - \frac{n!}{(b-a)^n}\left(f(b) - \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!}(b-a)^k \right),$$

i.e.

$$f(b) = \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!}(b-a)^k + \frac{f^{(n)}(c)}{n!}(b-a)^n$$

as desired. This completes the proof of Taylor's theorem. □

We can also give a second proof, based on the Cauchy mean-value theorem.

Second proof of Theorem 1. Let f be as in Theorem 1, and

$$F(x) = f(x) - \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!}(x-a)^k.$$

Let also

$$G(x) = (x-a)^n.$$

Then both F and G vanishes to order $(n-1)$ at a, in the sense that $F, F', F'', \ldots, F^{(n-1)}$ and $G, G', G'', \ldots, G^{(n-1)}$ all extends continuously to $[a, b]$, and the extended functions satisfy

$$F(a) = F'(a) = \cdots = F^{(n-1)}(a) = 0,$$

$$G(a) = G'(a) = \cdots = G^{(n-1)}(a) = 0.$$

Note also that $G', G'', \ldots, G^{(n)}$ all never vanishes on (a, b). Hence we may apply Cauchy's mean-value theorem n times: the first time we obtain

$$\frac{F(b)}{G(b)} = \frac{F(b) - F(a)}{G(b) - G(a)} = \frac{F'(c_1)}{G'(c_1)}$$

for some $c_1 \in (a, b)$. Next we can repeat this argument, on the interval $[a, c_1]$ instead of $[a, b]$: we then obtain

$$\frac{F'(c_1)}{G'(c_1)} = \frac{F'(c_1) - F'(a)}{G'(c_1) - G'(a)} = \frac{F''(c_1)}{G''(c_1)}$$

for some $c_2 \in (a, c_1)$. Repeating, we obtain $c_1, \ldots, c_n$ such that

$$a < c_n < c_{n-1} < \cdots < c_1 < b,$$

with

$$\frac{F(b)}{G(b)} = \frac{F'(c_1)}{G'(c_1)} = \frac{F''(c_2)}{G''(c_2)} = \cdots \frac{F^{(n)}(c_n)}{G^{(n)}(c_n)}.$$

In particular, setting $c = c_n$, we have $c \in (a, b)$, and

$$\frac{F(b)}{G(b)} = \frac{F^{(n)}(c)}{G^{(n)}(c)}.$$

This is equivalent to saying that

$$\frac{f(b) - \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!}(b-a)^k}{(b-a)^n} = \frac{f^{(n)}(c)}{n!},$$

which upon rearranging yields

$$f(b) = \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!}(b-a)^k + \frac{f^{(n)}(c)}{n!}(b-a)^n$$

as desired. This completes the second proof of Taylor's theorem. □