MATH1010A-H Assignment 4 Due date 13-11-2015

1. Refer to the respective pairs of functions and numbers below (separated by the semi-colon). Denote the function concerned by f, and the number by c. Find the degree-6 Taylor polynomial $T_{c,f,6}(x)$ of the function f about the point c.

   (a) $x^4 + x^2 + 1$; -2.

   (b) $\sin(3x)$; 0.

   (c) $\cos(x)$; $\frac{\pi}{4}$.

   (d) $e^{4x} + 2e^{-2x} + 3e^x$; 0.

   (e) $\sqrt{2+x}$; 1.

   (f) $\sqrt[3]{1+2x}$; 2.

   (g) $\frac{2}{4+9x^2}$; 0.

   (h) $\frac{2-x}{3+x}$; 1.

   (i) $\ln(e+t)$; 0.

   (j) $\ln(1-e^2t^2)$; 0.

   (k) $\cos(3x)\cos(2x)$; 0.

   (l) $\sin^2(4x) - \sin^2(3x)$; 0.

   Remark. In some of the above questions, it helps to simplify the expression of the function before calculating its Taylor polynomial.

2. Compute the degree 6 Taylor polynomials of

   (a) $\sec x$;

   (b) $\tan x$

   centered at 0.

   Hint. To compute the degree 6 Taylor polynomial of $\sec x$ centered at 0, we need to compute the derivatives of $\sec x$ at $x = 0$ up to order 6. The following tricks will help achieve this in a simpler manner.

   First, $\sec x$ is an even function, so if we differentiate $\sec x$ an odd order of times and evaluate at 0, we must get zero.

   Next, take the identity

   $$\sec x \cos x = 1, \tag{1}$$

   and differentiate it twice. Using Leibniz's rule (which we state towards the end of this question), one obtains the second derivative of $\sec x$ at $x = 0$.

   Now take again the identity (1), and differentiate it 4 times and 6 times respectively. Then using Leibniz's rule again, one obtains the 4th order and 6th order derivatives of $\sec x$ at $x = 0$.

   A similar trick works for tangent, since

   $$\tan x \cos x = \sin x.$$

   The Leibniz's Rule states:

   - Let $c \in \mathbb{R}$, and f, g be functions defined at and near the point c. Suppose each of f, g is n-times differentiable at c. Then the function $f \cdot g$ is n-times differentiable at c, and

   $$(f \cdot g)^{(n)}(c) = \sum_{j=0}^{n} \frac{n!}{(j!)[(n-j)!]} f^{(j)}(c) g^{(n-j)}(c).$$

   You may use this rule without proof.

   (It is a test of character if you apply brute force to directly differentiate the tangent function repeatedly.)

3. (a) Let $\alpha \in (0,1)$, and $x \in (0,1)$. Apply Taylor's Theorem with remainder of Lagrange form to show that

   $$\left| (1+x)^{\alpha} - 1 - \sum_{n=1}^{N} \frac{\alpha(\alpha-1)\cdot \ldots \cdot(\alpha-n+1)}{n!} x^n \right| \leq \frac{2\alpha x^{N+1}}{N+1}.$$

   (b) Hence find an approximation of $(1.2)^{0.2}$ within an error of 10^{-4}.

   (Hint: First look for a (sufficiently large) positive integer N which satisfies $\frac{2 \cdot 0.2 \cdot 0.2^{N+1}}{N+1} \leq \frac{1}{10^4}$.)

4. Evaluate the limits below. Where appropriate and convenient, you may apply L'Hopital's Rule.

(a) $\lim_{x\to 0} \frac{\sin 3x}{\sin 5x}$

(b) $\lim_{x\to 0} \frac{\sin^2 x}{1-\cos x}$

(c) $\lim_{x\to 0} \frac{2\sin x - \sin 2x}{x - \sin x}$

(d) $\lim_{x\to 0} \frac{1 - x\cot x}{x \sin x}$

(e) $\lim_{x\to 0} \frac{\sinh x - \sin x}{x(\cosh x - \cos x)}$

(f) $\lim_{x\to 0} \frac{\ln\cos 2x}{\ln\cos x}$

(g) $\lim_{x\to 0} \left(\frac{1}{x} - \frac{1}{e^x - 1}\right)$

(h) $\lim_{x\to 1} \left(\frac{1}{\ln x} - \frac{1}{x-1}\right)$

(i) $\lim_{x\to 0} \frac{e^x - x - 1}{\cosh x - 1}$

(j) $\lim_{x\to 0} \frac{e - (1+x)^{\frac{1}{x}}}{x}$

(k) $\lim_{x\to 0} \frac{2^x - 1}{x}$

(l) $\lim_{x\to 0^+} x^{\frac{1}{1+\ln x}}$

(m) $\lim_{x\to 1} x^{\frac{1}{1-x}}$

(n) $\lim_{x\to +\infty} \frac{\ln(2x^3 - 5x^2 + 3)}{\ln(4x^2 + x - 7)}$

(o) $\lim_{x\to +\infty} x \sin\left(\frac{1}{x}\right)$

(p) $\lim_{x\to +\infty} x\left(\frac{\pi}{2} - \tan^{-1} x\right)$

(q) $\lim_{x\to +\infty} x \ln\left(1 + \frac{3}{x}\right)$

(r) $\lim_{x\to +\infty} (e^x + x)^{\frac{1}{x}}$

(s) $\lim_{x\to 0} \frac{\cos x - e^{-x^2/2}}{x^4}$

(t) $\lim_{x\to 0} \frac{e^x \sin(x) - x(1+x)}{x^3}$

(u) $\lim_{x\to +\infty} x^{\frac{3}{2}}(\sqrt{x+1} + \sqrt{x-1} - 2\sqrt{x})$

(v) $\lim_{x\to +\infty} \left[x - x^2 \ln\left(1 + \frac{1}{x}\right)\right]$

5. Find the degree-20 Taylor polynomials of the following functions below about the point 0:

(a) $\sin(x^2)$

(b) $\sin(x^4)$

(c) $\sin(x^8)$

(d) $\cos(x^2) - \cos(x^4)$

(e) $\ln\left(\frac{1+x^4}{1+x^2}\right)$

You may use freely the following result:

- If f is infinitely differentiable at 0, and $g(x) = f(x^2)$, then g is infinitely differentiable at 0, with

$$g^{(m)}(0) = \begin{cases} 0 & \text{if } m \text{ is odd} \\ \frac{(2k)!}{k!} f^{(k)}(0) & \text{if } m \text{ is even and } m = 2k \text{ for some non-negative integer } k \end{cases}$$

(The potential math majors should find a proof of the above result!)