Cooperative games

Non-cooperative game

In non-cooperative game, the solution may not be a satisfactory result for the players.

Price war

		WC	
		Low	**High**
PN	**Low**	**2,2**	**5,0**
	High	**0,5**	**4,4**

	Payoffs
Nash equilibrium	(2,2)
Better result	(4,4)

Dating game

		Rachel	
		Football	**Drink**
Ross	**Football**	**10,5**	**0,0**
	Drink	**0,0**	**5,10**

In either of the Nash equilibriums, one of the players would not be satisfied.

Money sharing game

1. Five players put certain amount of money from $0 to $1,000 to a pool.
2. The total amount of money in the pool will be multiplied by 3.
3. The money in the pool is then distributed evenly to the players.

Money sharing game

	Ideal Situation	Nash Equilibrium
Strategy	$1,000	$0
Payoff	$2,000	$0

No one will put money to the pool because every dollar a player puts become 3 dollars but will share evenly with 5 players.

Environment protection

The money sharing game explains why every country is blaming others instead of putting more resources to environmental protection.

一蚊雞或無廣告世界盃

【明報18/4/2010】無綫、亞視在轉播世界盃的處理上與有線再次談不攏。

有線要求兩家免費台一元的版權費，但就要把有線世界盃賽事連廣告一齊播，… 這等於讓有線同時出賣無綫、亞視的廣告時間告時間送給有線。… 有線當然可以把廣告費大大提高。兩台當然不會應承，有線則可以說兩台不顧廣大觀眾利益，因這做法對觀眾有利，對有線更有利，只損害兩台收益。

無綫、亞視提出反建議，有線只需提供四場世界盃的主要賽事給兩台，而兩台則不會在這賽事中放任何廣告，即不利用世界盃來搵錢，只求讓更多觀眾可以收看。有線很快便拒絕了兩台這反建議。

筆者認為兩台可播世界盃的可能性愈來愈低，好看的反而是有線跟兩台互相過招，大家表面上都以觀眾利益作大前提，內裏當然是希望取得最大利益。到目前為止，雖然任何方案都是想更多人看到世界盃，卻沒一個可為雙方接受，問題當然不在觀眾利益之上。

World Cup broadcast

Additional payoff

additional commercial income

Pay TV proposal

- Put their commercial at Free TV
- Gain all additional income

Free TV proposal

- Do not put any commercial
- Abandon all additional income

三台達協議播放世界盃

【明報 27/4/2010(二）】有線電視終與兩間免費電視台，就轉播4場主要賽事達成協議，無線及亞視將於數碼頻道播放由有線提供的4場直播賽事連廣告。…

三個電視台昨日傍晚突然發表聲明，指「基於公眾利益」達成播放本屆世界盃賽事協議，… 一致感謝政府居中協助及斡旋。

有線曾去信兩台，提出只收取象徵式10元的轉播費用，但兩台必須播放有線的世界盃節目，包括廣告。兩台指有線的建議佔用的廣告時段，故不同意播廣告，如今由數碼頻道播放可算「各退一步」。

NBA談判徹底破裂（體育）

2011-11-15 歷時兩年半的NBA勞資協議談判遭遇重挫。球員工會拒絕資方提交的最新修訂提案，準備解散工會，以《反壟斷法》向資方提出訴訟。而NBA主席史坦就警告，如果工會不接受建議，資方的立場會轉趨強硬。

鑑於解散工會和動用法律手段解決勞資糾紛需要至少數個月，球員的決定很可能意味著2011至2012賽季整體報廢。如果真的如此，那將是NBA史上首次因停賽而斷送整個賽季。

NBA negotiation

NBA negotiation

美國NBA球季有望聖誕重開

2011-11-27美國NBA勞資談判出現曙光，勞資雙方經過最近一輪15小時的漫長談判，達成框架協議，常規賽有望在12月25日開始，但場數會由82場，縮減至66場。

Non-transferable utility

Cooperative game with non-transferable utility:

- A player cannot transfer its utility (payoff) to another player.
- The players may use joint strategy instead of using mixed strategy independently.

Joint strategy

Joint strategy:
Two players use varies pairs of strategies according to certain probabilities.

Examples:

1. Rock-scissors-paper:
   Using rock-rock with probability 0.7 and paper-scissors with probability 0.3.
2. Dating game:
   Watching soccer match with probability 0.1 and watching opera with probability 0.9.

Broadcasting rights game

Two broadcasting companies, NTV and CTV, bid for the exclusive broadcasting rights of a sports event. If both companies bid, NTV will win the bidding with a profit of $20 (million) and CTV will have no profit. If only NTV bids, there'll be a profit of $50 (million). If only CTV bids, there'll be a profit of $40 (million).

Broadcasting rights game

		CTV	
		Bid	Not
NTV	Bid	(20,0)	(50,0)
	Not	(0,40)	(0,0)

Bargaining problem

Nash proposed that a reasonable solution should satisfies the following axioms

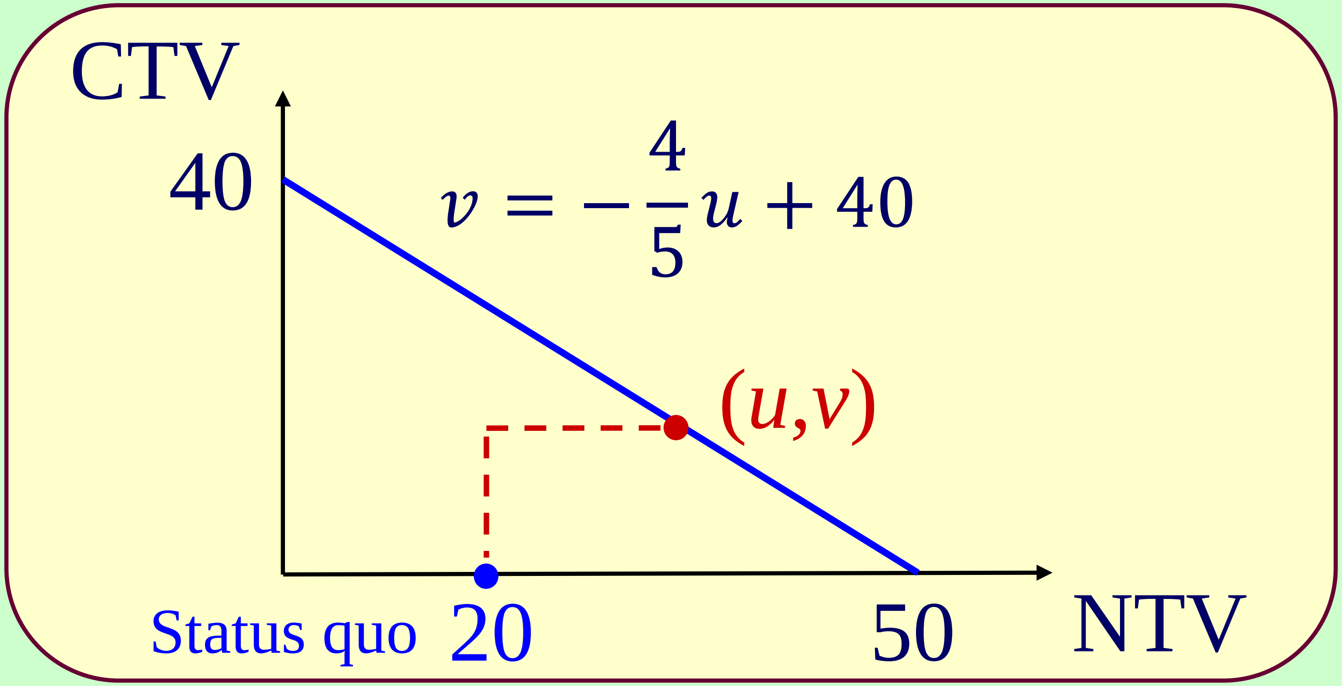
1. Pareto optimality
2. Independence of irrelevant alternatives
3. Invariant under linear transformation
4. Symmetry

Nash bargaining solution

Nash bargaining solution

Maximizing product of additional payoffs to the two players.

Broadcasting rights game



Product of additional payoffs $= (u - 20)v$

Broadcasting rights game

	NTV	CTV
Nash bargaining solution	Bidding 70% of the time	Bidding 30% of the time
Payoff (in million)	$35	$12
Additional payoff (in million)	$15	$12

Transferable utility

Cooperative game with transferable utility

1) 2-person game: Treat solution

2) N-person game: Core, Shapley value, …

Two-person cooperative games

		Colin	
		L	R
Rose	U	(100,0)	(-10,50)
	D	(20,10)	(10,-40)

The maximum total payoff is 100. How should the players split the total payoff if they want to cooperate?

Two-person cooperative games

There is no general rules that every player would or should follow. We are seeking for a fair solution: an outcome that will adequately represent the players' bargaining position, though not their bargaining abilities.

Threat matrix

Sum matrix:

$$S = R + C = \begin{pmatrix} 100 & 40 \\ 30 & -30 \end{pmatrix}$$

Threat matrix:

$$T = R - C = \begin{pmatrix} 100 & -60 \\ 10 & 50 \end{pmatrix}$$

Threat matrix

$$
\begin{array}{ccccc}
 & \begin{pmatrix} 100 & -60 \\ 10 & 50 \end{pmatrix} & \begin{array}{c} 160 \\ -40 \end{array} & \times & \begin{array}{c} 1/5 \\ 4/5 \end{array} \\
T = & & & & \\
 & \begin{array}{cc} 90 & -110 \end{array} & & & \\
 & \times & & & \\
 & \begin{array}{cc} \frac{11}{20} & \frac{9}{20} \end{array} & & &
\end{array}
$$

Threat strategies

Threat differential

The game value of the threat matrix is called the threat differential.

$$T = \begin{pmatrix} 100 & -60 \\ 10 & 50 \end{pmatrix}$$

Rose's threat strategy	Colin's threat strategy	Threat differential
(1/5,4/5)	(11/20,9/20)	28

Threat solution

The threat solution to a two-person cooperative game is the one where

1. The sum of the payoffs of the 2 players equals to the maximum entry of the sum matrix, and
2. The difference of the payoffs of the 2 players equals to the threat differential.

Threat solution

Rose should get 28 more than Colin. Let *x* and *y* be the amount that Rose and Colin get in the threat solution respectively, we have

$$\begin{cases} x+y=100 \\ x-y=28 \end{cases}$$

Thus

$$\begin{cases} x=\dfrac{100+28}{2}=64 \\ y=\dfrac{100-28}{2}=36 \end{cases}$$

Example 1

		Colin	
		L	R
Rose	U	(6,4)	(1,7)
	D	(3,2)	(3,0)

Example 1

$$\begin{pmatrix} (6,4) & (1,7) \\ (3,2) & (3,0) \end{pmatrix}$$

Nash equilibrium:

Rose's strategy	Colin's strategy	Payoff to Rose	Payoff to Colin
(2/5,3/5)	(2/5,3/5)	3	2.8

Threat differential

$$T = \begin{pmatrix} 2 & -6 \\ 1 & 3 \end{pmatrix}$$

Rose's threat strategy	Colin's threat strategy	Threat differential
(0.2,0.8)	(0.9,0.1)	1.2

Threat solution

The maximum total payoff is 10.
Therefore the threat solution is
Rose gets

$$\frac{10+1.2}{2}=5.6$$

and Colin gets

$$\frac{10-1.2}{2}=4.4$$

Example 2

		Colin	
		L	R
Rose	U	(2,0)	(5,8)
	D	(7,8)	(0,6)

Example 2

$$\begin{pmatrix} (2,0) & (5,8) \\ (7,8) & (0,6) \end{pmatrix}$$

Nash equilibria:

Rose's strategy	Colin's strategy	Payoff to Rose	Payoff to Colin
(1/5,4/5)	(1/2,1/2)	3.5	6.4
(1,0)	(0,1)	5	8
(0,1)	(1,0)	7	8

Threat differential

$$T = \begin{pmatrix} 2 & -3 \\ -1 & -6 \end{pmatrix}$$

Rose’s threat strategy	Colin’s threat strategy	Threat differential
(1,0)	(0,1)	-3

Threat solution

The maximum total payoff is 15.

Therefore the threat solution is

Rose gets

$$\frac{15+(-3)}{2}=6$$

and Colin gets

$$\frac{15-(-3)}{2}=9$$

Example 3

		Colin	
		L	R
Rose	U	(5,0)	(8,4)
	D	(9,7)	(4,3)

Example 3

$$\begin{pmatrix} (5,0) & (8,4) \\ (9,7) & (4,3) \end{pmatrix}$$

Nash equilibria:

Rose's strategy	Colin's strategy	Payoff to Rose	Payoff to Colin
(1/2,1/2)	(1/2,1/2)	6.5	3.5
(1,0)	(0,1)	8	4
(0,1)	(1,0)	9	7

Threat matrix

Sum matrix:

$$S = R + C = \begin{pmatrix} 5 & 12 \\ 16 & 7 \end{pmatrix}$$

Threat matrix:

$$T = R - C = \begin{pmatrix} 5 & 4 \\ 2 & 1 \end{pmatrix}$$

Threat differential

$$T = \begin{pmatrix} 5 & 4 \\ 2 & 1 \end{pmatrix}$$

Rose's threat strategy	Colin's threat strategy	Threat differential
(1,0)	(0,1)	4

Threat solution

The maximum total payoff is 16.
Therefore the threat solution is
Rose gets

$$\frac{16+4}{2}=10$$

and Colin gets

$$\frac{16-4}{2}=6$$

Threat solution vs Nash

	Payoff to Rose	Payoff to Colin
Mixed Nash equilibrium	6.5	3.5
Non-Pareto pure Nash equil.	8	4
Pareto pure Nash equil.	9	7
Threat solution	10	6

It is not always good to cooperate.

N-person cooperative games

Suppose there are *n*-persons, P_1, P_2, P_3, … P_n, in a game. A coalition is a collection of players.

Example: $n = 3$

There are 7 coalitions

$\{P_1\}$, $\{P_2\}$, $\{P_3\}$, $\{P_{1,} P_2\}$, $\{P_{2,} P_3\}$, $\{P_{1,} P_3\}$,$\{P_{1,} P_{2,} P_3\}$

Counter coalition

Let $S \subset \{P_1, P_2, P_3, \ldots, P_n\}$ be a coalition. The counter-coalition S^c of S is the coalition formed by the collection of players not in S.

S	$\{P_1\}$,	$\{P_3\}$	$\{P_{1,} P_3\}$
S^c	$\{P_{2,} P_3\}$	$\{P_{1,} P_2\}$	$\{P_2\}$

Characteristic function

For each coalition S, we assign a value $v(S)$ which is the maximum payoff for the coalition S. The function v is called the characteristic function.

Characteristic function

The value of the characteristic function $v(S)$ can be computed by solving the 2-coalition non-cooperative game between S and S^c.

Characteristic function

The characteristic function *v* of an *n*-person game satisfies

$$v(S \cup T) \geq v(S) + v(T)$$

for any disjoint coalitions *S* and *T*.

Coalitions and characteristic function

- **Set of players:** $N = \{1, 2, 3, \cdots, n\}$
- **Coalition: A collection of players in *N* is called a coalition, i.e., *S* is a coalition if** $S \subset N$
- **For any coalition *S*, define**

  ***v*(*S*) = max. utility *S* can get without the cooperation of S^c**

  ***v* is called the characteristic function.**
- **Let *S* and *T* be two disjoint coalitions. Then**

$$v(S \cup T) \geq v(S) + v(T)$$

Imputation

Definition: (x_1,x_2,x_3,…,x_n) is called an imputation if

1. $x_k \geq v(\{k\})$ **for any k = 1,2,3,…,n.**

2. $x_1 + x_2 + x_3 + \cdots + x_n = v(N)$

Remarks:

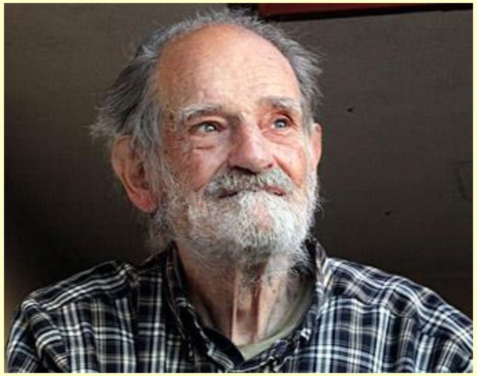
- **Here x_k is the possible payoff of player k.**
- **An imputation is a reasonable way to distribute the payoffs.**
- **Imputation of cooperative game is usually not unique.**

Lloyd Stowell Shapley

- **Born in 1923**
- **His father Harlow Shapley is known for determining the position of the Sun in the Milky Way Galaxy**

Lloyd Stowell Shapley

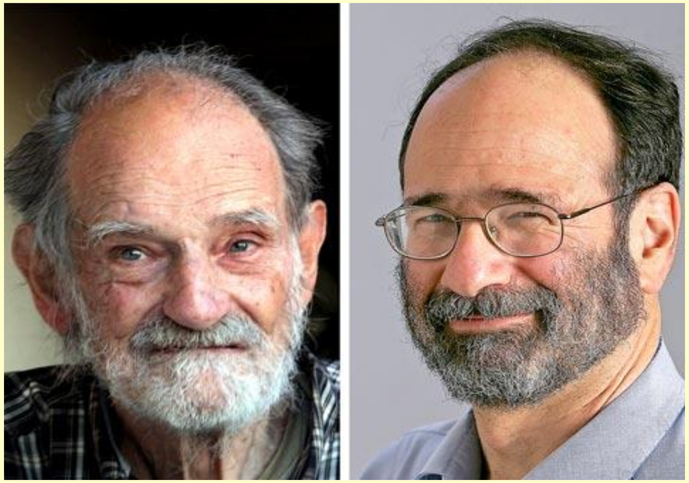
- **Drafted when he was a student at Harvard in 1947**



- **Served in the Army in Chengdu, China and received the Bronze Star decoration for breaking the Japanese weather code**

Nobel Prize in Economic 2012

- **A value for *n*-person Games (1953)**
- **College Admissions and the Stability of Marriage (with Davis Gale 1962)**
- **Awarded Nobel Memorial Prize in Economic Sciences with Alvin Elliot Roth in 2012**



Shapley Roth

Nobel Prize in Economic 2012

This year's Prize concerns a central economic problem: how to match different agents as well as possible. For example, students have to be matched with schools, and donors of human organs with patients in need of a transplant. How can such matching be accomplished as efficiently as possible? What methods are beneficial to what groups? The prize rewards two scholars who have answered these questions on a journey from abstract theory on stable allocations to practical design of market institutions.

Shapley value

The Shapley value of player *k* is defined as

$$\phi_k = \sum_{S \subset N} \frac{(|S|-1)!(n-|S|)!}{n!} \delta(k, S)$$

where

$$\delta(k,S) = v(S) - v(S \setminus \{k\})$$

is the contribution of player *k* to coalition *S*.

Shapley's value of player *k* is the average contribution of player *k* to all orders of coalitions.

2-person cooperative game

When the number of players is 2,

$$\phi_1 = \frac{v(\{1\}) + v(\{1,2\}) - v(\{2\})}{2}$$
$$= v(\{1\}) + \frac{v(\{1,2\}) - \left(v(\{1\}) + v(\{2\})\right)}{2}$$

$$\phi_2 = \frac{v(\{2\}) + v(\{1,2\}) - v(\{1\})}{2}$$
$$= v(\{2\}) + \frac{v(\{1,2\}) - \left(v(\{1\}) + v(\{2\})\right)}{2}$$

2-person cooperative game

For 2-person games, the players share evenly the additional payoff gained by cooperation.

Two-person games

		II	
		L	R
I	U	(100,0)	(-10,50)
	D	(20,10)	(10,-40)

Solutions	I's strategy	II's strategy	Payoff to I	Payoff to II
Nash equil.	(1/2,1/2)	(1/5,4/5)	12	5
Treat solution	(1/5,4/5)	(9/20,11/20)	64	36

Two-person games

		II	
		L	R
I	U	(100,0)	(-10,50)
	D	(20,10)	(10,-40)

Coalition	$v(S)$
{1}	12
{2}	5
{1,2}	100

Two-person games

$$\phi_1 = v(\{1\}) + \frac{v(\{1,2\}) - (v(\{1\}) + v(\{2\}))}{2}$$

$$= 12 + \frac{100 - (12 + 5)}{2}$$

$$= 53.5$$

$$\phi_2 = v(\{2\}) + \frac{v(\{1,2\}) - (v(\{1\}) + v(\{2\}))}{2}$$

$$= 5 + \frac{100 - (12 + 5)}{2}$$

$$= 46.5$$

Two-person games

		II	
		L	R
I	U	(100,0)	(-10,50)
	D	(20,10)	(10,-40)

Solutions	I's strategy	II's strategy	Payoff to I	Payoff to II
Nash equil.	(1/2,1/2)	(1/5,4/5)	12	5
Treat solution	(1/5,4/5)	(9/20,11/20)	64	36
Shapley	-	-	53.5	46.5

Restaurant coupon

Suppose Rose has a coupon

RAINBOW CAFÉ

20% OFF FOR SINGLE

50% OFF FOR COUPLE

Rose invited Colin to dinner at Rainbow café. They plan to spend $100 each before discount. How should they split the bill?

Restaurant coupon

Coalition	Original	Need to pay	$v(S)$
{R}	100	80	20
{C}	100	100	0
{R,C}	200	100	100

Restaurant coupon

$$\phi_R = 20 + \frac{100 - 20}{2} = 60$$

$$\phi_C = 0 + \frac{100 - 20}{2} = 40$$

Rose should pay $40 and Colin should pay $60.

Build an airport

Two cities Rose and Colin want to build an airport somewhere near the mid-point of the two cities. They may choose whether to join the building project or not.

Build an airport

The cost and benefit (in billion dollars) to the two cities of the project are listed as follows

Build	Rose's Cost	Colin's Cost	Rose's Benefit	Colin's Benefit
Together	8	8	18	13
Rose	16	3	21	9
Colin	5	11	12	15
None	0	0	-6	-3

Build an airport

		Colin	
		Yes	No
Rose	Yes	(10,5)	(5, 6)
	No	(7,4)	(-6,-3)

There is a unique Nash equilibrium: Rose plays 'Yes' and Colin plays 'No'. The payoffs are 5 and 6 respectively.

Build an airport

		Colin	
		Yes	No
Rose	Yes	(10,5)	(5, 6)
	No	(7,4)	(-6,-3)

Coalition	$v(S)$
{R}	5
{C}	6
{R,C}	15

Additional payoff

= 10+5 – (5+6) = 4

Build an airport

$$\phi_1 = v(\{1\}) + \frac{v(\{1,2\}) - \left(v(\{1\}) + v(\{2\})\right)}{2}$$

$$= 5 + \frac{4}{2}$$

$$= 7$$

$$\phi_2 = v(\{2\}) + \frac{v(\{1,2\}) - \left(v(\{1\}) + v(\{2\})\right)}{2}$$

$$= 6 + \frac{4}{2}$$

$$= 8$$

Build an airport

		Colin	
		Yes	No
Rose	Yes	(10,5)	(5, 6)
	No	(7,4)	(-6,-3)

Solutions	Payoff to Rose	Payoff to Colin
Nash equilibrium	5	6
Shapley’s value	7	8

Shapley value for 3-person games

Order	$S \setminus \{1\}$	S	$\delta(1,S)$
123	{ }	{1}	$v(\{1\})$
132	{ }	{1}	$v(\{1\})$
213	{2}	{1,2}	$v(\{1,2\}) - v(\{2\})$
231	{2,3}	{1,2,3}	$v(\{1,2,3\}) - v(\{2,3\})$
312	{3}	{1,3}	$v(\{1,3\}) - v(\{3\})$
321	{2,3}	{1,2,3}	$v(\{1,2,3\}) - v(\{2,3\})$

Shapley value for 3-person games

When the number of players is 3,

$$\phi_1 = \frac{1}{6}\begin{pmatrix} 2v(\{1\}) + \big(v(\{1,2\}) - v(\{2\})\big) \\ + \big(v(\{1,3\}) - v(\{3\})\big) + 2\big(v(\{1,2,3\}) - v(\{2,3\})\big) \end{pmatrix}$$

Assume that

$$v(\{1\}) = v(\{2\}) = v(\{3\}) = 0$$

Then

$$\phi_1 = \frac{v(\{1,2\}) + v(\{1,3\}) - 2v(\{2,3\}) + 2v(\{1,2,3\})}{6}$$

Shapley value for 3-person games

Shapley's values for 3-person cooperative game:

Assume that $v(\{1\}) = v(\{2\}) = v(\{3\}) = 0$

$$\phi_1 = \frac{v(\{1,2\}) + v(\{1,3\}) - 2v(\{2,3\}) + 2v(\{1,2,3\})}{6}$$

$$\phi_2 = \frac{v(\{2,1\}) + v(\{2,3\}) - 2v(\{1,3\}) + 2v(\{1,2,3\})}{6}$$

$$\phi_3 = \frac{v(\{3,1\}) + v(\{3,2\}) - 2v(\{1,2\}) + 2v(\{1,2,3\})}{6}$$

Sharing taxi fare

Andy, Betty and Cindy, want to go to City One, Tai Wai and Tsuen Wan respectively from CUHK by taxi. The taxi fares are given in the following table.

Destination	Fare
City One	$50
Tai Wai	$60
Tsuen Wan	$120

Sharing taxi fare

Sharing taxi fare

However, they can save some money by hiring a taxi together and sharing the taxi fare.

Destination (S)	Fare	Save ($v(S)$)
City One & Tai Wai	$80	$50+$60-$80=$30
City One & Tsuen Wan	$150	$50+$120-$150=$20
Tai Wai & Tsuen Wan	$130	$60+$120-$130=$50
All 3 places	$160	$50+$60+$120-$160=$70

Sharing taxi fare

Player's contribution to orders of coalitions

Order	Player 1 (Andy) contribution
123	0
132	0
213	$v(\{1,2\})$
231	$v(\{1,2,3\}) - v(\{2,3\})$
312	$v(\{1,3\})$
321	$v(\{1,2,3\}) - v(\{2,3\})$

Sharing taxi fare

The additional payoff of Andy is

$$\phi_1 = \frac{v(\{1,2\}) + v(\{1,3\}) - 2v(\{2,3\}) + 2v(\{1,2,3\})}{6}$$

$$= \frac{30 + 20 - 2(50) + 2(70)}{6}$$

$$= 15$$

Andy should pay \$50 - \$15 = \$35

Sharing taxi fare

The additional payoff of Betty is

$$\phi_2 = \frac{v(\{1,2\}) + v(\{2,3\}) - 2v(\{1,3\}) + 2v(\{1,2,3\})}{6}$$

$$= \frac{30 + 50 - 2\left(20\right) + 2(70)}{6}$$

$$= 30$$

Betty should pay \$60 - \$30 = \$30

Sharing taxi fare

The additional payoff of Cindy is

$$\phi_3 = \frac{v(\{1,3\}) + v(\{2,3\}) - 2v(\{1,2\}) + 2v(\{1,2,3\})}{6}$$

$$= \frac{20 + 50 - 2\left(30\right) + 2(70)}{6}$$

$$= 25$$

Cindy should pay \$120 - \$25 = \$95

Sharing taxi fare

Player	Destination	Original fare	Save	New fare
Andy	City One	$50	$15	$35
Betty	Tai Wai	$60	$30	$30
Cindy	Tsuen Wan	$120	$25	$95