MMATH 5220 Assignment 1 sol

1 (a) $(1+z+\cdots+z^n)\cdot z = z+z^2+\cdots+z^{n+1}$

$\therefore (1+z+\cdots+z^n)\cdot(1-z) = 1+z+\cdots+z^n - z - \cdots - z^{n+1}$

$= 1-z^{n+1}$

$\therefore (1+z+\cdots+z^n) = \dfrac{1-z^{n+1}}{1-z}$

(b) Assume $z = e^{i\theta}$

we have $1+e^{i\theta}+e^{2i\theta}+\cdots+e^{ni\theta} = \dfrac{1-e^{i(n+1)\theta}}{1-e^{i\theta}}$

$$= \frac{e^{-i\frac{\theta}{2}}\left(1-e^{i(n+1)\theta}\right)}{e^{-i\frac{\theta}{2}}\left(1-e^{i\theta}\right)} = \frac{e^{-i\frac{\theta}{2}} - e^{i(n+\frac{1}{2})\theta}}{e^{-i\frac{\theta}{2}} - e^{i\frac{\theta}{2}}}$$

$$= \frac{\cos\frac{\theta}{2} - i\sin\frac{\theta}{2} - \cos(n+\frac{1}{2})\theta - i\sin(n+\frac{1}{2})\theta}{\cos\frac{\theta}{2} - i\sin\frac{\theta}{2} - \cos\frac{\theta}{2} - i\sin\frac{\theta}{2}}$$

$$= \frac{\cos\frac{\theta}{2} - \cos(n+\frac{1}{2})\theta - i\sin\frac{\theta}{2} - i\sin(n+\frac{1}{2})\theta}{-2i\sin\frac{\theta}{2}}$$

Compare the real part, we get

$$1+\cos\theta+\cos 2\theta+\cdots+\cos n\theta = \frac{1}{2} + \frac{\sin(n+\frac{1}{2})\theta}{2\sin\frac{\theta}{2}}$$

2 no the statement is not true.

Assume $z_1 = z_2 = -1 = e^{i\pi}$

$$\mathrm{Log}(z_1 z_2) = \mathrm{Log}(1) = 0$$

$$\mathrm{Log}\, z_1 + \log z_2 = \mathrm{Log}\, 1 + i\pi + \mathrm{Log}\, 1 + i\pi = 2\pi i$$

3. (a) if $c=0$, we have $T(z) = \dfrac{az+b}{d}$ and $ad \neq 0$

$ad \neq 0 \Rightarrow a \neq 0$ & $d \neq 0$

$\forall\ \varepsilon > 0$ if $|z| > \dfrac{|d|}{|a|}\cdot \varepsilon + \left|\dfrac{b}{a}\right|$

Then $$|T(z)| = \frac{|az+b|}{|d|} \geq \frac{|az| - |b|}{|d|} > \frac{|d|\varepsilon + |b| - |b|}{|d|} = \varepsilon$$

$$\therefore \lim_{z\to\infty} T(z) = \infty$$

(b) ① First we have $\displaystyle\lim_{z\to\infty} \frac{b}{z} = 0$ and $\displaystyle\lim_{z\to\infty} \frac{d}{z} = 0$, Then

$$\lim_{z\to\infty} T(z) = \lim_{z\to\infty} \frac{az+b}{cz+d} = \lim_{z\to\infty} \frac{a+\frac{b}{z}}{c+\frac{d}{z}} = \frac{a}{c}$$

② $$\lim_{z\to -\frac{d}{c}} az+b = a\frac{-d}{c} + b = \frac{-ad+bc}{c} \neq 0$$

$$\lim_{z\to -\frac{d}{c}} cz+d = 0$$

$$\therefore \lim_{z\to -\frac{d}{c}} \frac{az+b}{cz+d} = \infty$$

4 $\because 0 \le |f(z)\cdot g(z)| \le |f(z)|\cdot M$ in some neighborhood of z_0

$$\therefore 0 \le \lim_{z\to z^0} |f(z)\cdot g(z)| \le \lim_{z\to z^0} |f(z)|\cdot M = 0$$

$$\therefore \lim_{z\to z^0} |f(z)\cdot g(z)| = \lim_{z\to z^0} f(z)g(z) = 0$$

5. Assume $z = x+iy$, then we have

$f(z) = \bar{z} = x - iy = u(x,y) + v(x,y)i$, where $u(x,y) = x$, $v(x,y) = -y$

$\frac{\partial u}{\partial x} = 1$ $\frac{\partial v}{\partial y} = -1$ $\therefore \frac{\partial u}{\partial x} \neq \frac{\partial v}{\partial y}$ $\therefore f'(z)$ does not exist at any point

6. Suppose $f(z) = u(x,y) + iv(x,y)$

$\Rightarrow \overline{f(z)} = u(x,y) - iv(x,y)$ $\quad \forall z \in D$

Because they are both analytic at z, we have

$$\begin{cases} \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \end{cases} \text{ and } \begin{cases} \frac{\partial u}{\partial x} = -\frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} = \frac{\partial v}{\partial x} \end{cases}$$

$$\therefore \frac{\partial u}{\partial x} = \frac{\partial u}{\partial y} = \frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} = 0$$

$\therefore$ u, v are constant $\Rightarrow f$ is constant in D