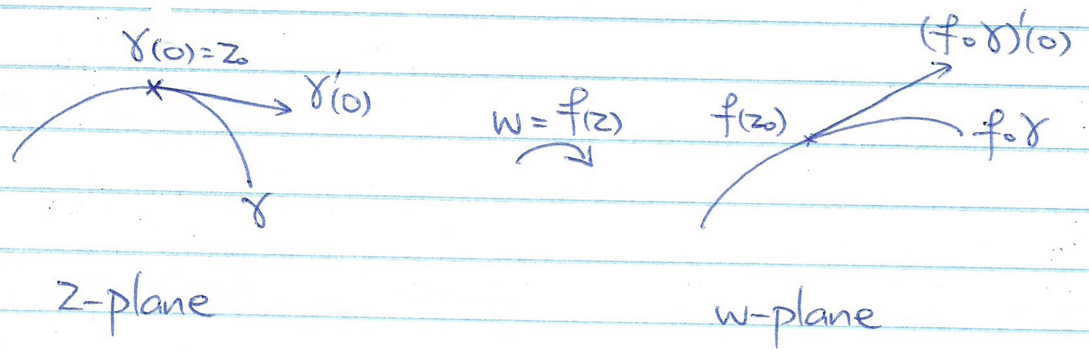
# Conformal Mappings.

Let $\gamma : (-\varepsilon, \varepsilon) \to \mathbb{C}$ $\quad \gamma(t) = x(t) + i y(t)$

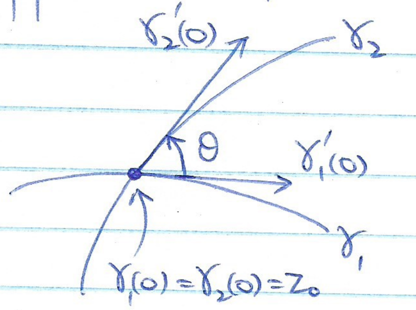
$$\gamma'(0) = \lim_{t \to 0} \frac{\gamma(t) - \gamma(0)}{t - 0} = x'(0) + i y'(0)$$

Let $\gamma(0) = z_0$, $f : \mathbb{C} \to \mathbb{C}$ is complex differentiable at $z_0$,
then $(f \circ \gamma)'(0) = f'(z_0) \cdot \gamma'(0)$ (Chain rule)




Note: If $f'(z_0) \neq 0$, then $\arg(f \circ \gamma)'(0) = \arg(f'(z_0)) + \arg(\gamma'(0))$

Suppose $\gamma_1, \gamma_2 : (-\varepsilon, \varepsilon) \to \mathbb{C}$ and $\gamma_1(0) = \gamma_2(0) = z_0$.




(oriented) angle between $\gamma_1$ and $\gamma_2$ (at $z_0$)
$:=$ (oriented) angle between $\gamma_1'(0)$ and $\gamma_2'(0)$ (at $z_0$)

A function $f$, with continuous partial derivatives, is said to be conformal at $z_0$ if
(oriented) angle between $\gamma_1$ and $\gamma_2$ (at $z_0$)
$=$ (oriented) angle between $f \circ \gamma_1$ and $f \circ \gamma_2$ (at $f(z_0)$)

A function is said to be conformal in a domain $D$ if it is conformal at every point in $D$.

Theorem : $f$ is complex differentiable at $z_0$ and $f'(z_0) \neq 0$ if and only if $f$ is conformal at $z_0$.

$(\Rightarrow)$ $\arg(f\circ\gamma_2)'(0) - \arg(f\circ\gamma_1)'(0)$

$= (\arg f'(z_0) + \arg \gamma_2'(0)) - (\arg f'(z_0) + \arg \gamma_1'(0))$

$= \arg \gamma_2'(0) - \arg \gamma_1'(0)$

$(\Leftarrow)$ Choose $\gamma_1(t) = z_0 + t$, $\gamma_2(t) = z_0 + ti$

$\gamma_1'(0) = 1$, $\gamma_2'(0) = i$

$\therefore$ angle between $\gamma_1$ and $\gamma_2$ at $z_0 = \frac{\pi}{2}$.

Note : If we write $f(z) = f(x+iy) = u(x,y) + iv(x,y)$, $z_0 = x_0 + iy_0$

$(f\circ\gamma_1)(t) = u(x_0+t, y_0) + iv(x_0+t, y_0)$

$\therefore (f\circ\gamma_1)'(0) = u_x(x_0,y_0) + iv_x(x_0,y_0)$

Similarly, $(f\circ\gamma_2)'(0) = u_y(x_0,y_0) + iv_y(x_0,y_0)$

- $|(f\circ\gamma_1)'(0)|^2 = u_x^2(x_0,y_0) + v_x^2(x_0,y_0)$
  $|(f\circ\gamma_2)'(0)|^2 = u_y^2(x_0,y_0) + v_y^2(x_0,y_0)$

$\therefore (f\circ\gamma_1)'(0)$ and $(f\circ\gamma_2)'(0)$ have same length at $f(z_0)$ and by assumption, angle between $f\circ\gamma_1$ and $f\circ\gamma_2$ = angle between $\gamma_1$ and $\gamma_2$ at $z_0$

$\therefore i(f\circ\gamma_1)'(0) = (f\circ\gamma_2)'(0)$

$i(u_x(x_0,y_0) + iv_x(x_0,y_0)) = u_y(x_0,y_0) + iv_y(x_0,y_0)$

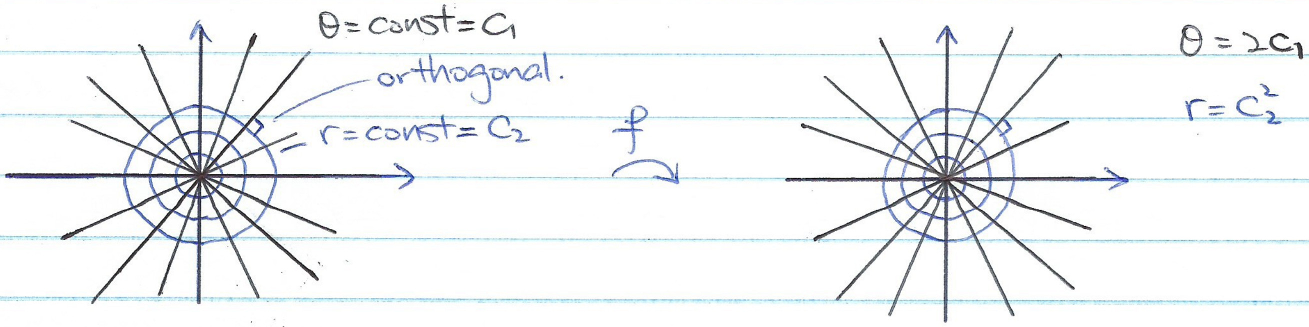
$\therefore$ CR-eq$^n$ is satisfied at $z_0 = x_0 + iy_0$.

(+ $f$ has continuous partial derivative at $z_0$)

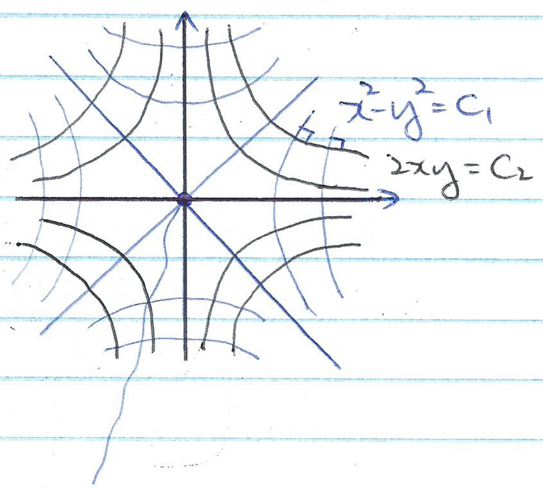
$\therefore f$ is analytic at $z_0$ and also

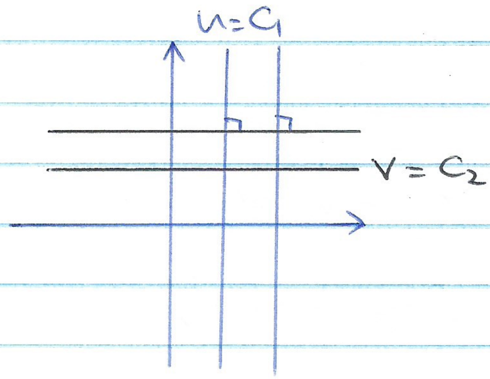
$0 \neq |(f\circ\gamma_1)'(0)| = |f'(z_0)||\gamma_1'(0)| = |f'(z_0)|$

e.g. $f(z) = z^2$




z-plane                                   w-plane




$$\begin{aligned} f(z) &= f(x+iy) \\ &= (x+iy)^2 \\ &= (x^2-y^2) + i(2xy) \end{aligned}$$

$$\therefore\ u(x,y) = x^2 - y^2$$
$$v(x,y) = 2xy$$

level curves $u = \text{const}$ and $v = \text{const}$ are orthogonal everywhere except $z = 0$ (Not conformal at $z = 0$)

Not surprising!
Note: $f'(0) = 0$

$g : \mathbb{D} \to \mathbb{C}$ defined by $g(z) = \dfrac{z-a}{1-\bar{a}z}$, where $|a|<1$
and $\mathbb{D}$ is the open unit disk $= \{|z|<1\}$.

Claim: $g$ is conformal, bijective map from $\mathbb{D}$ to $\mathbb{D}$.

- $1-\bar{a}z = 0$
  $\Rightarrow z = \frac{1}{\bar{a}} \Rightarrow |z| = \left|\frac{1}{\bar{a}}\right| > 1$
  $\therefore 1-\bar{a}z \neq 0 \quad \forall z \in \mathbb{D}$
  $\Big\}$ Well-defined.

- $g'(z) = \dfrac{(1-\bar{a}z)-(z-a)(-\bar{a})}{(1-\bar{a}z)^2}$
  $= \dfrac{1-|a|^2}{(1-\bar{a}z)^2} \neq 0$
  $\Big\} \Rightarrow g'(z) \neq 0 \quad \forall z \in \mathbb{D}$
  $\Downarrow$
  $g$ is conformal in $\mathbb{D}$

- If $|z|<1$

$$(1-|z|^2)(1-|a|^2) > 0$$
$$1+|z|^2|a|^2 > |z|^2+|a|^2$$
$$1-a\bar{z}-\bar{a}z+|z|^2|a|^2 > |z|^2-a\bar{z}-\bar{a}z+|a|^2$$
$$(1-a\bar{z})(1-\bar{a}z) > (\bar{z}-\bar{a})(z-a)$$
$$|1-\bar{a}z|^2 > |z-a|^2$$
$$1 > \left|\frac{z-a}{1-\bar{a}z}\right|^2 = |g(z)|^2$$

$\therefore$ Image of $g \subseteq \mathbb{D}$

- $g : \mathbb{D} \to \mathbb{D}$ is one-to-one (injective), onto (surjective)
  (Exercise)

More elegant way:

- $g(z)$ is a fractional linear transformation.

A fractional linear transformation is a function
$f(z) = \dfrac{az+b}{cz+d}$ where $a,b,c,d \in \mathbb{C}$ and $ad-bc \neq 0$

If $f$ is a fractional linear transformation, then

- $f$ is conformal
- $f$ maps circles (in Riemann sphere) to circles (in Riemann sphere) / maps circles or straight lines in $\mathbb{C}$ to circles or straight lines in $\mathbb{C}$.
- $f$ is a composition of
  dilations $z \mapsto az$
  translations $z \mapsto a+z$
  inversions $z \mapsto \frac{1}{z}$
  $\therefore$ bijective

Now $g(z) = \dfrac{z-a}{1-\bar{a}z}$ maps $\{|z|=1\}$ to $\{|z|=1\}$ and $g(a)=0$,

So $g: \mathbb{D} \to \mathbb{D}$ is conformal and bijective.

Furthermore, if $c \neq 0$, define $f(-\frac{d}{c}) = \infty$, $f(\infty) = \frac{a}{c}$

if $c = 0$, define $f(\infty) = \infty$

then $f(z) = \frac{az+b}{cz+d}$ extends to the Riemann sphere $\hat{\mathbb{C}}$

and $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is biholomorphic.

Check: $\mathrm{Aut}(\hat{\mathbb{C}}) = \{ f(z) = \frac{az+b}{cz+d} : a,b,c,d \in \mathbb{C},\ ad-bc \neq 0 \}$
is a group (group multiplication = composition of maps)

Matrix representation: $f(z) = \frac{az+b}{cz+d} \rightsquigarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$ad - bc \neq 0$ $\qquad$ $\det \neq 0$

$\therefore$ $\pi : GL(2,\mathbb{C}) \longrightarrow \mathrm{Aut}(\hat{\mathbb{C}})$ defines a surjective group homomorphism.

$$\pi\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \frac{az+b}{cz+d} = f(z)$$

$$\ker \pi = \left\{ \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \,\middle|\, \lambda \in \mathbb{C},\ \lambda \neq 0 \right\} = \mathbb{C}^*$$

$$\therefore \mathrm{Aut}(\hat{\mathbb{C}}) \cong GL(2,\mathbb{C}) / \ker \pi = \underset{\substack{\| \\ PGL(2,\mathbb{C})}}{GL(2,\mathbb{C}) / \mathbb{C}^*}$$

Theorem (Schwarz Lemma)

Let $f(z)$ be an analytic function for $|z|<1$.
Suppose $|f(z)| \le 1$ for all $|z|<1$ and $f(0)=0$,
then $|f(z)| \le |z|$ for all $|z|<1$.
If the equality holds at some point $z_0 \neq 0$, then
$f(z) = \lambda z$ for some $\lambda \in \mathbb{C}$ with $|\lambda| = 1$.

proof: Talyor expansion of $f$ at $z=0$:

$$f(z) = \cancel{f(0)}^{\,0} + \underbrace{f'(0)z + \frac{f''(0)}{2!}z^2 + \cdots}_{= \, zg(z)}$$

$g(z)$ is analytic.

Apply Max. principle to $g(z)$ on $\{|z|=r\}$, $r<1$

$$|g(z)| = \left|\frac{f(z)}{z}\right| = \frac{|f(z)|}{r} \le \frac{1}{r}$$

$\therefore$ $|g(z)| \le \frac{1}{r}$ $\forall$ $|z|<r$

Let $r \to 1$, $|g(z)| \le 1$ $\forall$ $|z|<1$ (i.e. $\left|\frac{f(z)}{z}\right| \le 1$
$\Rightarrow$ $|f(z)| \le |z|$ $\forall$ $z \neq 0$
$z=0$, trivial)

If $|f(z_0)| = |z_0|$ for some $z_0 \neq 0$,

$$|g(z_0)| = \left|\frac{f(z_0)}{z_0}\right| = 1 \Rightarrow g \text{ is a constant (CR-eq}^{\underline{n}} \text{ helps)}$$

$g = \lambda$ and so $|\lambda| = 1$

$\therefore$ $f(z) = \lambda z$ and we write $\lambda = e^{i\varphi}$ for some $0 \le \varphi < 2\pi$.

**Lemma:** If $g : \mathbb{D} \to \mathbb{D}$ is a conformal, bijective map such that $g(0) = 0$, then $g(z) = e^{i\varphi} z$ for some $0 \le \varphi < 2\pi$.

Schwarz Lemma $\overset{\text{to } g}{\Longrightarrow}$ $|g(z)| \le |z|$ $\qquad w = g(z)$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad z = g^{-1}(w)$

$\overset{\text{to } g^{-1}}{\Longrightarrow}$ $|g^{-1}(w)| \le |w|$

$\Downarrow$

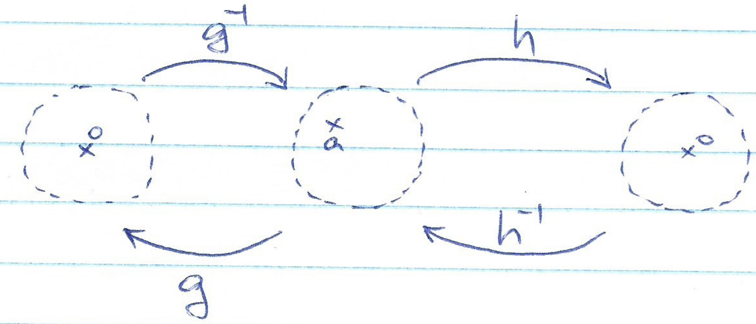
$|z| \le |g(z)|$

$\therefore |g(z)| = |z| \Rightarrow \left|\frac{g(z)}{z}\right| = 1 \Rightarrow \frac{g(z)}{z}$ is a constant.

$\therefore g(z) = e^{i\varphi} z$

**Theorem:** Every conformal, bijective map from $\mathbb{D}$ to $\mathbb{D}$ is of the form $f(z) = e^{i\varphi} \dfrac{z-a}{1-\bar{a}z}$, where $|a| < 1$ and $0 \le \varphi < 2\pi$

proof: Let $h : \mathbb{D} \to \mathbb{D}$ be an arbitrary conformal bijective map.




Suppose $h(a) = 0$ for some $a \in \mathbb{D}$

Let $g(z) = \dfrac{z-a}{1-\bar{a}z}$, then $g(a) = 0$

$\therefore h \circ g^{-1}(0) = 0$

Apply the previous lemma, $h \circ g^{-1}(z) = e^{i\varphi} z$

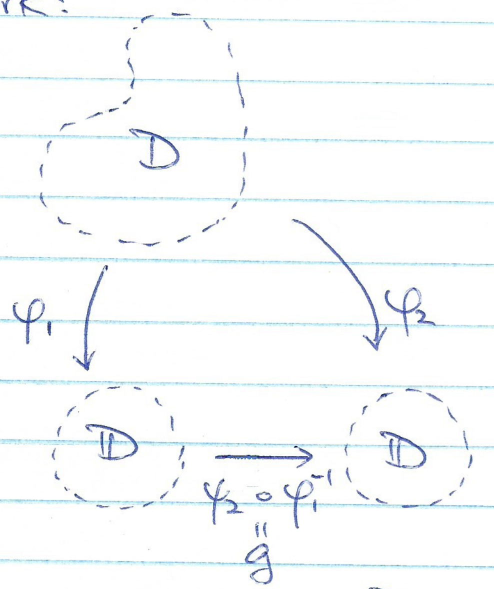
$h(z) = e^{i\varphi} g(z)$

## Riemann Mapping Theorem

If $D$ is a simply connected domain in $\mathbb{C}$, and $D \neq \mathbb{C}$, then there exists a bijective conformal map $\varphi : D \to \mathbb{D}$.

Remark:

1)




Suppose $\varphi_1, \varphi_2 : D \to \mathbb{D}$ are conformal and bijective then $\varphi_2 \circ \varphi_1^{-1} : \mathbb{D} \to \mathbb{D}$ is conformal and bijective

$$\therefore \varphi_2 \circ \varphi_1^{-1}(z) = e^{i\varphi} \frac{z-a}{1-\bar{a}z} = g(z)$$

$$\varphi_2(z) = g \circ \varphi_1(z)$$

$\therefore$ Once we can write down $\varphi_1$, we determine every $\varphi : D \to \mathbb{D}$

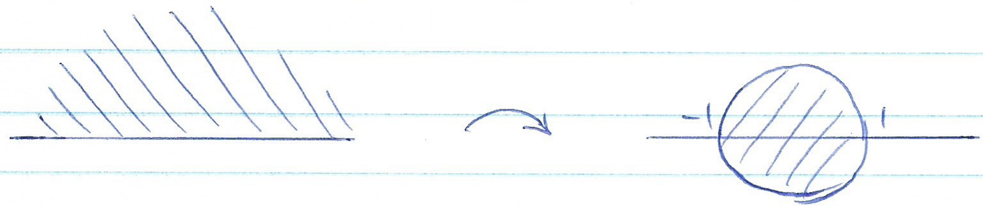
2) Determine all bijective conformal maps $f : D \to D$, If we can write down $\varphi : D \to \mathbb{D}$

$$\begin{array}{ccc} D & \xrightarrow{f} & D \\ \varphi^{-1} \uparrow & & \downarrow \varphi \\ \mathbb{D} & \xrightarrow[g]{} & \mathbb{D} \end{array}$$

Suppose $f : D \to D$ is bijective and conformal, then $\varphi \circ f \circ \varphi^{-1} : \mathbb{D} \to \mathbb{D}$

$$\therefore \varphi \circ f \circ \varphi^{-1}(z) = g(z) \Rightarrow f = \varphi^{-1} \circ g \circ \varphi$$

## Upper Half plane $\mathbb{H}$ :

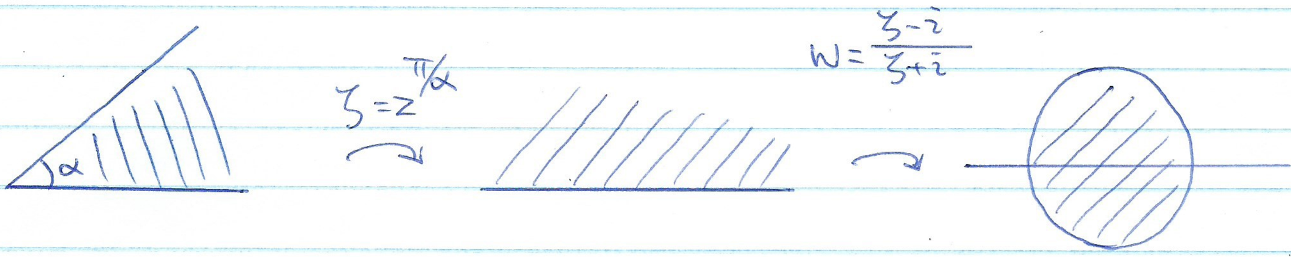



z-plane w-plane

$$w = \frac{z-i}{z+i}$$

Check : If $z = x+iy$ , $y>0$ , then $|w|<1$

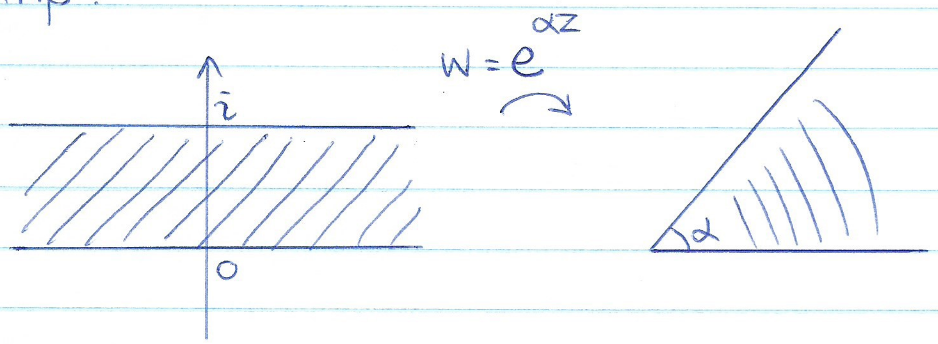
## Sector




z-plane $\zeta$-plane w-plane.

$$w = \frac{z^{\pi/\alpha} - i}{z^{\pi/\alpha} + i}$$

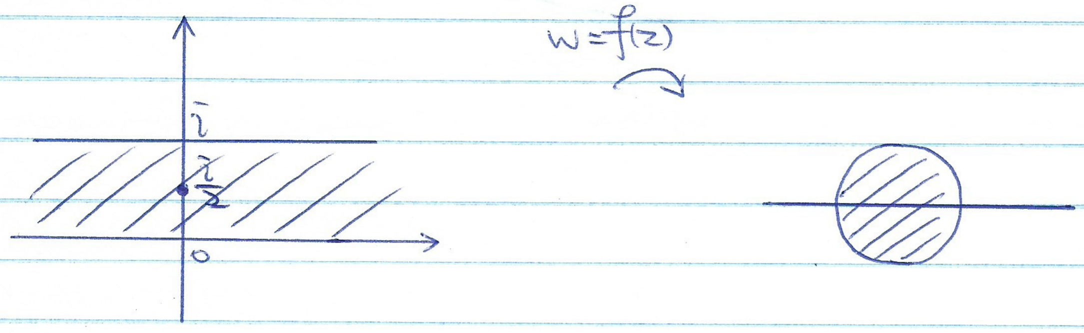
## Strip.




In particular, choose $\alpha = \pi$, it becomes an upper half plane.

z-plane $\zeta$-plane

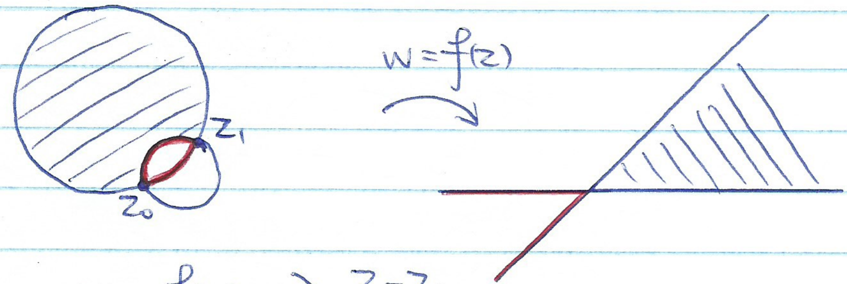
$\zeta = e^{\alpha(x+iy)} = e^{\alpha x} \cdot e^{i\alpha y}$ $\quad 0 < y < 1$, $\quad 0 < \alpha y < \alpha$




z-plane w-plane

$$w = f(z) = \frac{e^{\pi z} - i}{e^{\pi z} + i}$$

## Lunar Domain




$$w = f(z) = \lambda \frac{z - z_0}{z - z_1}$$

$$f(z_0) = 0$$
$$f(z_1) = \infty$$