Extending Cauchy-Goursat Theorem :

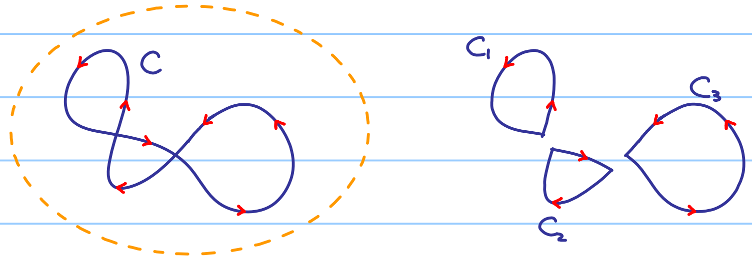
Theorem : If f is differentiable throughout a simply connected domain D, then

$$\int_C f(z)dz = 0$$

for every closed contour C lying in D.

proof :

case 1 : C intersects itself a finite number of times.

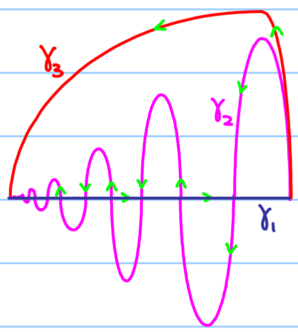


$$\int_C f(z)dz = \int_{C_1} f(z)dz + \int_{C_2} f(z)dz + \int_{C_3} f(z)dz$$

$= 0 + 0 + 0$ (Apply Cauchy-Goursat Theorem to each C_i)

$= 0$

case 2 : C intersects itself an infinite number of times.



$C = \gamma_1 \cup \gamma_2$

$$\int_C f(z)dz = \int_{\gamma_1} f(z)dz + \int_{\gamma_2} f(z)dz$$

$= \left(\int_{\gamma_1} f(z)dz + \int_{\gamma_3} f(z)dz\right) + \left(\int_{\gamma_2} f(z)dz - \int_{\gamma_3} f(z)dz\right)$

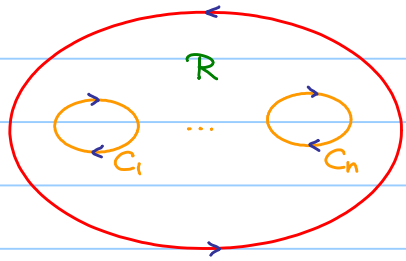
$= 0 + 0$ (Apply Cauchy-Goursat Theorem to each C_i)

$= 0$

Corollary :

If f is differentiable throughout a simply connected domain D, then f must have an antiderivative in D.

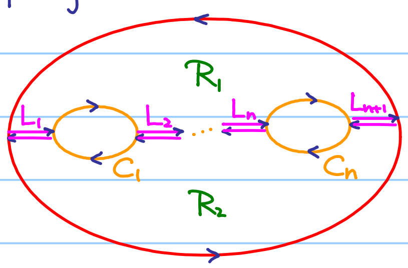
Theorem : Let C and C_k $(k=1,2,\cdots n)$ be simple closed contours such that



$\partial R = C \cup C_1 \cup \cdots \cup C_k$

If f is differentiable in an open neighborhood containing $\bar{R}$,
then $\int_C f(z)dz + \sum_{k=1}^{n} \int_{C_k} f(z)dz = 0$ OR : $\int_{\partial R} f(z)dz = 0$

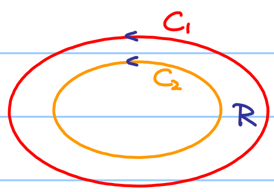
proof :



$\int_{\partial R} f(z)dz = \int_{\partial R_1} f(z)dz + \int_{\partial R_2} f(z)dz$

$= 0$ Note, R_1 and R_2 are simply connected.

Direct consequence : If f is differentiable on $\bar{R}$, then

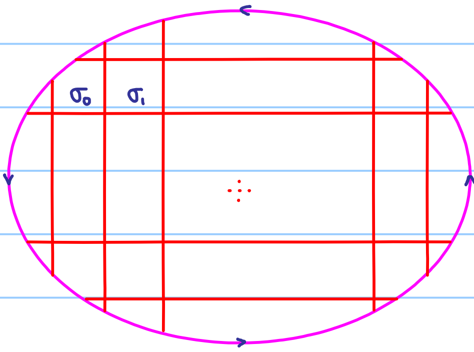


$\int_{C_1} f(z)dz = \int_{C_2} f(z)dz$

Cauchy-Goursat Theorem :

Let R be a simply connected domain with boundary C.
If a function f is differentiable in an open neighborhood containing $\bar{R}$, then $\int_C f(z)\,dz = 0$.

proof :

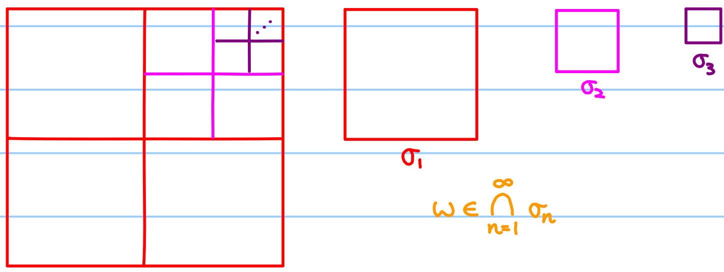


Use a grid paper to subdivide $\bar{R}$ into σ_j's

Lemma: $\forall \varepsilon > 0$, $\exists$ subdivision s.t. $\forall \sigma_j$, $\exists z_j \in \operatorname{int} \sigma_j$ s.t.

$$\left| \frac{f(z) - f(z_j)}{z - z_j} - f'(z_j) \right| < \varepsilon \quad \forall z \in \sigma_j \text{ with } z \neq z_j .$$

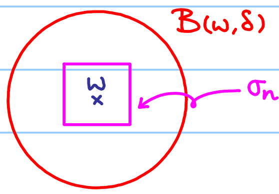
proof : Suppose the contrary, let $\varepsilon > 0$,



$\exists$ a sequence of σ_i s.t. $\forall z_j \in \sigma_j$, $\exists \check{z}_j \in \sigma_j$ s.t. $\left| \frac{f(\check{z}_j) - f(z_j)}{\check{z}_j - z_j} - f'(z_j) \right| \geq \varepsilon$

but f is differentiable at ω, $\exists \delta > 0$ s.t. $\left| \frac{f(z) - f(\omega)}{z - \omega} - f'(\omega) \right| < \varepsilon \quad \forall\, 0 < |z - \omega| < \delta$

(Contradiction)



$$\int_C f(z)\,dz = \sum_{j=1}^{N} \int_{\partial \sigma_j} f(z)\,dz .$$

$$\left| \int_C f(z)\,dz \right| = \left| \sum_{j=1}^{N} \int_{\partial \sigma_j} f(z)\,dz \right| \leq \sum_{j=1}^{N} \underbrace{\left| \int_{\partial \sigma_j} f(z)\,dz \right|}_{\text{estimate}}$$

Let $\delta_j(z) = \begin{cases} \dfrac{f(z)-f(z_j)}{z-z_j} - f'(z_j) & \text{if } z \neq z_j \\ 0 & \text{if } z = z_j \end{cases}$ on σ_j

Note: δ_j is continuous on σ_j and $|\delta_j(z)| < \varepsilon$

Then $f(z) = f(z_j) - z_j f'(z_j) + f'(z_j) z + (z-z_j)\delta_j(z)$

$$\int_{\partial\sigma_j} f(z)\,dz = \underbrace{\int_{\partial\sigma_j} f(z_j) - z_j f'(z_j)\,dz}_{=0} + \underbrace{\int_{\partial\sigma_j} f'(z_j) z\,dz}_{=0} + \int_{\partial\sigma_j} (z-z_j)\delta_j(z)\,dz$$

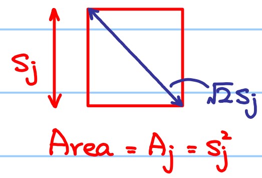
By Cauchy-Goursat Theorem

$$|(z-z_j)\delta_j(z)| \leq |z-z_j|\cdot|\delta_j(z)| \leq \sqrt{2}s_j\varepsilon$$

ML-estimate:

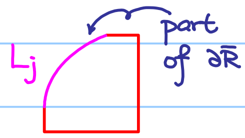
if σ_j is a square,

$$\left|\int_{\partial\sigma_j} f(z)\,dz\right| \leq 4s_j \cdot \sqrt{2}s_j\varepsilon = 4\sqrt{2}A_j\varepsilon$$



if σ_j is a part of a square,

$$\left|\int_{\partial\sigma_j} f(z)\,dz\right| \leq (4s_j + L_j)\cdot\sqrt{2}s_j\varepsilon = 4\sqrt{2}A_j\varepsilon + \sqrt{2}L_j s_j\varepsilon$$



$$\left|\int_C f(z)\,dz\right| = \left|\sum_{j=1}^{N}\int_{\partial\sigma_j} f(z)\,dz\right| \leq \sum_{j=1}^{N}\left|\int_{\partial\sigma_j} f(z)\,dz\right|$$

$$\leq \left[\sum_{j=1}^{N}(4\sqrt{2}A_j + \sqrt{2}L_j s_j)\right]\cdot\varepsilon$$

Theorem :

Suppose f is continuous on a domain D (NOT necessary to be simply connected)

The following statements are equivalent :

1) f has an antiderivative F in D ;

2) the integrals of $f(z)$ along contours lying entirely in D and extending from any fixed points z_1 to any fixed point z_2 all have the same value ;

3) the integrals of $f(z)$ along any closed contours lying entirely in D all have value zero.

proof : ((2) $\Leftrightarrow$ (3) trivial, omitted)

(1) $\Rightarrow$ (2) :

If C is a contour from z_1 to z_2, $C = C_1 \cup C_2 \cup \cdots \cup C_n$, where C_i's are differentiable arcs.

$\gamma(t)$ is a parametrization of C for $a \le t \le b$,

and $a = t_0 < t_1 < \cdots < t_{n-1} < t_n = b$, $\gamma|_{[t_{i-1}, i]}$ is C_i

$\frac{d}{dt} F(\gamma(t)) = F'(\gamma(t)) \cdot \gamma'(t) = f(\gamma(t)) \cdot \gamma'(t) \quad \forall\, t \in [a,b]$ except $t_0, t_1, \cdots, t_n$.

$$\int_C f(z)\,dz = \sum_{i=1}^{n} \int_{C_i} f(z)\,dz$$

$$= \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} f(\gamma(t)) \cdot \gamma'(t)\,dt$$

$$= \sum_{i=1}^{n} \big[F(\gamma(t)) \big]_{t_{i-1}}^{t_i}$$

$$= F(\gamma(b)) - F(\gamma(a))$$

$$= F(z_2) - F(z_1)$$ (which is independent from C but endpoints z_1 and z_2.)

(2) $\Rightarrow$ (1) :

By assumption, we are able to define $F(z) = \int_{z_0}^{z} f(z)\,dz$.

(We want to show $F'(z) = f(z)$.)

Let $\varepsilon > 0$,

f is cont at $z \Rightarrow \exists\, \delta > 0$ s.t.

$|f(s) - f(z)| < \varepsilon \quad \forall\, |s - z| < \delta$

Then if $0<|\Delta z|<\delta$,

$$\frac{F(z+\Delta z)-F(z)}{\Delta z} - f(z) = \frac{1}{\Delta z}\int_z^{z+\Delta z} f(s)-f(z)\,dz$$

$$\left|\frac{F(z+\Delta z)-F(z)}{\Delta z} - f(z)\right| = \frac{1}{|\Delta z|}\left|\int_z^{z+\Delta z} f(s)-f(z)\,dz\right|$$

Choosing line segment

ML-estimate

$$\le \frac{1}{|\Delta z|}\,\varepsilon|\Delta z|$$

$$= \varepsilon$$

Note:

$$\int_z^{z+\Delta z} ds = \Delta z$$

$$\int_z^{z+\Delta z} f(z)\,ds = f(z)\Delta z$$

$$f(z) = \frac{1}{\Delta z}\int_z^{z+\Delta z} f(z)\,ds$$

$$\therefore \lim_{\Delta z\to 0}\frac{F(z+\Delta z)-F(z)}{\Delta z} - f(z) = 0 \qquad \text{i.e. } F'(z)=f(z).$$

e.g. $f(z)=\frac{1}{z}$ is differentiable on $\mathbb{C}^*$ which is NOT simply connected.

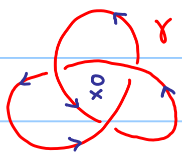
Recall: If $\gamma(t) = \cos t + i\sin t = e^{it}$ $\quad 0\le t\le 2\pi$ (unit circle)

$$\int_\gamma f(z)\,dz = 2\pi i \ne 0$$

$\therefore f$ has NO antiderivative on $\mathbb{C}^*$

However, by restricting $f(z)$ on $\mathbb{C}\setminus(-\infty,0]$ (which is simply connected)
Then $f(z)$ has an antiderivative $F(z) = \operatorname{Log} z$.

e.g. If γ is



then $\int_\gamma \frac{1}{z^2}\,dz = 0$,
as $f(z)=\frac{1}{z^2}$ has an antiderivative $F(z)=\frac{1}{z}$ on $\mathbb{C}^*$ and γ lies in $\mathbb{C}^*$.

IV) Cauchy Integral Formula

Theorem: Let C be a simple closed contour, taken in positive sense,

and R is the bounded domain with $\partial R = C$.

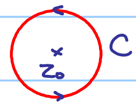
If f is differentiable on $\overline{R}$ and $z_0 \in R$, then

$$f(z_0) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-z_0}\,dz$$

Idea:

Ex: If C_0 is the positively oriented circle centered at z_0 with radius r,

show that $\displaystyle\int_{C_0}(z-z_0)^n\,dz = \begin{cases} 0 & \text{if } n \geq 0 \\ 2\pi i & \text{if } n=-1.\end{cases}$ (f is diff. everywhere in this case.)



Pretend: $f(z) = f(z_0) + f'(z_0)(z-z_0) + \frac{f''(z_0)}{2!}(z-z_0)^2 + \cdots$

$$\frac{f(z)}{z-z_0} = \frac{f(z_0)}{z-z_0} + f'(z_0) + \frac{f''(z_0)}{2!}(z-z_0) + \cdots$$

$$\int_C \frac{f(z)}{z-z_0}\,dz = \int_{C_0}\frac{f(z)}{z-z_0}\,dz$$

$$= \underbrace{\int_{C_0}\frac{f(z_0)}{z-z_0}\,dz}_{=\,2\pi i f(z_0)} + \underbrace{\int_{C_0} f'(z_0)\,dz}_{=\,0} + \underbrace{\int_{C_0}\frac{f''(z_0)}{2!}(z-z_0)\,dz}_{=\,0} + \cdots$$

proof: Let $\varepsilon > 0$,

f is diff at $z_0 \Rightarrow \exists\,\delta > 0$ s.t. $|f(z)-f(z_0)| < \varepsilon \quad \forall\, |z-z_0| < \delta$

choose ρ s.t. $0<\rho<\delta$, then

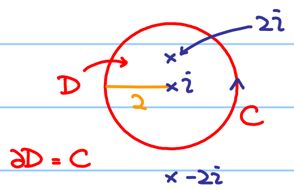
$|f(z)-f(z_0)| < \varepsilon \quad \forall\, |z-z_0| = \rho$.

$\therefore \left|\frac{f(z)-f(z_0)}{z-z_0}\right| < \frac{\varepsilon}{\rho}$

$$\left|\int_C \frac{f(z)}{z-z_0}\,dz - 2\pi i f(z_0)\right| = \left|\int_{C_\rho}\frac{f(z)-f(z_0)}{z-z_0}\,dz\right| \leq \frac{\varepsilon}{\rho}2\pi\rho = 2\pi\varepsilon$$

ε can be arbitrarily small $\Rightarrow \displaystyle\int_C \frac{f(z)}{z-z_0}\,dz - 2\pi i f(z_0) = 0$

e.g. $g(z) = \frac{1}{z^2+4}$, find $\int_C g(z)\,dz$.



$\partial D = C$

$\bar{D} = D \cup \partial D$

$g(z) = \frac{1}{z^2+4} = \frac{1}{(z-2i)(z+2i)} = \frac{f(z)}{z-2i}$ where $f(z) = \frac{1}{z+2i}$

Note: $f(z)$ is analytic on $\bar{D}$.

$\therefore \int_C g(z)\,dz = \int_C \frac{f(z)}{z-2i}\,dz = 2\pi i\, f(2i) = \frac{\pi}{2}$

Cauchy Integral Formula.

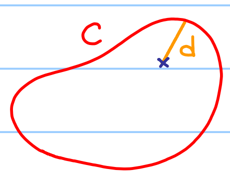
Let C be a simple closed contour, taken in positive sense, and R is the bounded domain with $\partial R = C$, $z \in R$

Cauchy Integral formula: $f(z) = \frac{1}{2\pi i}\int_C \frac{f(s)}{s-z}\,ds$

Want to show: $f'(z) = \frac{1}{2\pi i}\int_C \frac{f(s)}{(s-z)^2}\,ds$

$$\frac{f(z+\Delta z)-f(z)}{\Delta z} = \frac{1}{2\pi i}\int_C \left(\frac{1}{s-z-\Delta z} - \frac{1}{s-z}\right)\frac{f(s)}{\Delta z}\,ds$$

$$= \frac{1}{2\pi i}\int_C \frac{f(s)}{(s-z-\Delta z)(s-z)}\,ds$$



d: shortest distance from z to C.

$$\therefore \frac{f(z+\Delta z)-f(z)}{\Delta z} - \frac{1}{2\pi i}\int_C \frac{f(s)}{(s-z)^2}\,ds = \frac{\Delta z}{2\pi i}\int_C \frac{f(s)}{(s-z-\Delta z)(s-z)^2}\,ds$$

$$\left|\frac{f(z+\Delta z)-f(z)}{\Delta z} - \frac{1}{2\pi i}\int_C \frac{f(s)}{(s-z)^2}\,ds\right| \le \frac{|\Delta z| ML}{(d-|\Delta z|)d^2}$$

$$\therefore \lim_{\Delta z\to 0}\frac{f(z+\Delta z)-f(z)}{\Delta z} = \frac{1}{2\pi i}\int_C \frac{f(s)}{(s-z)^2}\,ds$$

for $s \in C$,

$|s-z| \ge d$

$|s-z-\Delta z| \ge \big||s-z|-|\Delta z|\big| \ge d-|\Delta z|$

$|f(s)| \le M$ for some $M>0$.

(Why? $\because$ f is cont. and C is cpt.)

L = length of C.

Apply the same technique:

Want to show $f''(z) = \frac{2}{2\pi i}\int_C \frac{f(s)}{(s-z)^3}\,ds$

$$\frac{f'(z+\Delta z)-f'(z)}{\Delta z} = \frac{1}{2\pi i}\int_C \frac{2(s-z)-\Delta z}{(s-z-\Delta z)^2(s-z)^2} f(s)\,ds$$

$$\underbrace{\frac{f'(z+\Delta z)-f'(z)}{\Delta z} - \frac{2}{2\pi i}\int_C \frac{f(s)}{(s-z)^3}\,ds}_{(*)} = \frac{1}{2\pi i}\int_C \frac{3(s-z)\Delta z - 2(\Delta z)^2}{(s-z-\Delta z)^2(s-z)^2} f(s)\,ds$$

Ex: Show (*) tends to as $\Delta z \to 0$.

Repeating the process:

Theorem:

If f is differentiable in a domain D, then it can be differentiated infinitely many times in D.

In general, if $z \in D$, $f^{(n)}(z) = \frac{n!}{2\pi i}\int_C \frac{f(s)}{(s-z)^{n+1}}\,ds$

where C is a simple closed contour in D and z lies in interior of C.

Corollary:

If f is differentiable in a domain D, then f' is continuous in D.

i.e. f is analytic in D!

Corollary:

If a function $f(z) = f(x+iy) = u(x,y) + iv(x,y)$ is analytic at a point $z = x+iy$, then the functions u and v have continuous partial derivatives of all orders at the point (x,y).

Theorem:

If a function is continuous throughout a domain D and if $\int_C f(z)\,dz = 0$ for every closed contour C lying in D, ($\Rightarrow$ antiderivative of f exists, $F'(z) = f(z) \Rightarrow F$ is analytic in D)

then f is analytic in D.

e.g. (f has an antiderivative in D $\not\Leftarrow$ f is analytic in D)

Consider $f(z) = \frac{1}{z}$,

f is analytic in $D = \mathbb{C}^*$ and $f'(z) = -\frac{1}{z^2}$

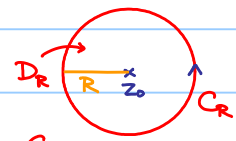
but f has NO antiderivative.

However, if D is simply connect, the converse statement is true!

$\therefore$ f has an antiderivative in $D \Leftrightarrow f$ is analytic in D

V) Liouville's Theorem and Fundamental Theorem of Algebra

Let C_R be the circle $|z-z_0| = R$, oriented in positive sense.



Let f be a function analytic on $\bar{D}_R$.

$\partial D_R = C_R$

By Cauchy Integral Formula,

$\bar{D}_R = D_R \cup \partial D_R$

$$f^{(n)}(z_0) = \frac{n!}{2\pi i}\int_{C_R} \frac{f(z)}{(z-z_0)^{n+1}}\,dz \qquad (n = 1, 2, \cdots)$$

Furthermore, if f is bounded on $\bar{D}_R$, i.e. $|f(z)| \le M_R \quad \forall z \in \bar{D}_R$

Then, we have the following estimates:

$$\begin{aligned} |f^{(n)}(z_0)| &= \left| \frac{n!}{2\pi i}\int_{C_R} \frac{f(z)}{(z-z_0)^{n+1}}\,dz \right| \\ &\le \frac{n!}{2\pi} \frac{M_R}{R^{n+1}} \cdot 2\pi R \qquad (\text{ML-estimate}) \\ &= \frac{n!M_R}{R^n} \end{aligned}$$

Further generalize:

Let f be an entire function, i.e. analytic everywhere.

and suppose f is a bound function, $|f(z)| \le M \quad \forall z \in \mathbb{C}$.

For every C_R, we have

$$\begin{aligned} |f'(z_0)| &= \left| \frac{1}{2\pi i}\int_{C} \frac{f(z)}{(z-z_0)^2}\,dz \right| \\ &\le \frac{1}{2\pi} \frac{M}{R^2} \cdot 2\pi R \qquad (\text{ML-estimate}) \\ &= \frac{M}{R} \end{aligned}$$

Since R can be arbitrarily large, $f'(z_0) = 0$

$\therefore$ $f'(z) = 0 \quad \forall z \in \mathbb{C}$.

Theorem:

If f is entire and bounded, then $f(z)$ is constant throughout the plane.

Fundamental Theorem of Algebra :

Any polynomial $P(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n$ ($a_i \in \mathbb{C}$ and $a_n \neq 0$) of degree n ($n \geq 1$) has at least one zero, i.e. $\exists z_0 \in \mathbb{C}$ s.t. $P(z_0) = 0$.

proof : Prove by contradiction,

Suppose $P(z) \neq 0$ $\forall z \in \mathbb{C}$, then $f(z) = \frac{1}{P(z)}$ is an entire function.

Claim : $f(z)$ is bounded ($\Rightarrow f$ is constant $\Rightarrow$ Contradiction)

To show $f(z)$ is bounded, it suffices to show f is bounded outside a large disk.

Note : $f(z) = \frac{1}{P(z)} = \frac{1}{z^n\left(a_n + \frac{a_{n-1}}{z} + \cdots + \frac{a_0}{z^n}\right)}$

Consider $|z| \geq R$, we can choose a large R such that

$$\frac{|a_n|}{2} \geq \left|\frac{a_{n-1}}{z} + \cdots + \frac{a_0}{z^n}\right|$$

$$\therefore \left|a_n + \frac{a_{n-1}}{z} + \cdots + \frac{a_0}{z^n}\right| \geq \left||a_n| - \left|\frac{a_{n-1}}{z} + \cdots + \frac{a_0}{z^n}\right|\right| \geq \frac{|a_n|}{2}$$

$$|f(z)| = \left|\frac{1}{z^n\left(a_n + \frac{a_{n-1}}{z} + \cdots + \frac{a_0}{z^n}\right)}\right| \leq \frac{2}{|a_n| R^n}$$

Direct consequence : Any polynomial of degree $n \geq 1$, can be factorized as a product of linear factors.

VI) Maximum Moduli of Functions

Lemma : Suppose that $f(z)$ is analytic throughout a neighborhood $|z - z_0| < \varepsilon$ of a point z_0. If $|f(z)| \leq |f(z_0)|$ for each point in that neighborhood, then $f(z)$ has the constant value $f(z_0)$ throughout the neighborhood.

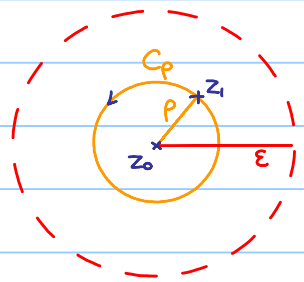
$$f(z_0) = \frac{1}{2\pi i}\int_{C_\rho} \frac{f(z)}{z - z_0}\,dz$$

Let $z = z_0 + \rho e^{i\theta}$

$0 \leq \theta \leq 2\pi$

$$= \frac{1}{2\pi}\int_0^{2\pi} f(z_0 + \rho e^{i\theta})\,d\theta$$

$dz = i\rho e^{i\theta}\,d\theta$



$$|f(z_0)| \le \frac{1}{2\pi}\int_0^{2\pi} |f(z_0+\rho e^{i\theta})|\, d\theta$$

$$\le \frac{1}{2\pi}\int_0^{2\pi} |f(z_0)|\, d\theta$$

$$= |f(z_0)|$$

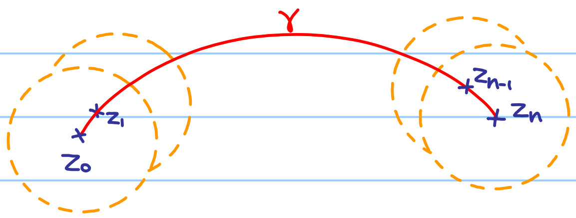
$$\therefore\ 2\pi |f(z_0)| = \int_0^{2\pi} |f(z_0+\rho e^{i\theta})|\, d\theta$$

$$\int_0^{2\pi} \underbrace{|f(z_0)| - |f(z_0+\rho e^{i\theta})|}_{\geq 0}\, d\theta = 0$$

$$\Rightarrow |f(z_0)| = |f(z_0+\rho e^{i\theta})|$$

In particular, $f(z_0) = f(z_1)$

Theorem: Suppose that $f(z)$ is analytic throughout a domain D and suppose $z_0 \in D$. If $|f(z)| \le |f(z_0)|$ for each point in that domain D, then $f(z)$ has the constant value $f(z_0)$ throughout the domain.



$f(z_0) = f(z_1) = \cdots = f(z_n)$ Applying the lemma to each disk!

(Remark: Why we can cover the curve by finitely many disks.)