Solution to Homework 3, MMAT5000

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(1) **Proof:** Let $(X_1, \rho_1), (X_2, \rho_2)$ be metric spaces. Define $X = X_1 \times X_2$ and $\rho_E, \rho_{\max}, \rho_{sum} : X \times X \to \mathbb{R}$ by:

$$\rho_E((x_1, x_2), (y_1, y_2)) = \sqrt{\rho_1^2(x_1, y_1) + \rho_2^2(x_2, y_2)},$$

$$\rho_{\max}((x_1, x_2), (y_1, y_2)) = \max\{\rho_1(x_1, y_1), \rho_2(x_2, y_2)\},$$

$$\rho_{sum}((x_1, x_2), (y_1, y_2)) = \rho_1(x_1, y_1) + \rho_2(x_2, y_2).$$

For all $(x, y) \in X$, we have

$$\rho_{\max}(x, y) \leq \rho_E(x, y) \leq \rho_{sum}(x, y) \leq 2\rho_{\max}(x, y).$$

- For each $U \in \mathfrak{T}_{\rho_E}$: $\forall (x_1, x_2) \in U$, $\exists \delta > 0$, such that $(y_1, y_2) \in U$ whenever $(y_1, y_2) \in X$ and $\rho_E((x_1, x_2), (y_1, y_2)) < \delta$. For such $(x_1, x_2) \in U$, $\exists \delta_1 = \delta > 0$, if $\rho_{sum}((x_1, x_2), (y_1, y_2)) < \delta_1$, then $\rho_E((x_1, x_2), (y_1, y_2)) \leq \rho_{sum}((x_1, x_2), (y_1, y_2)) < \delta_1 = \delta$, and hence $(y_1, y_2) \in U$. In other words, $\forall (x_1, x_2) \in U$, $\exists \delta_1 > 0$ such that $(y_1, y_2) \in U$ whenever $(y_1, y_2) \in X$ and $\rho_{sum}((x_1, x_2), (y_1, y_2)) < \delta_1$. This implies that $U \in \mathfrak{T}_{\rho_{sum}}$, and therefore $\mathfrak{T}_{\rho_E} \subseteq \mathfrak{T}_{\rho_{sum}}$.
- For each $U \in \mathfrak{T}_{\rho_{sum}}$: $\forall (x_1, x_2) \in U$, $\exists \delta > 0$, such that $(y_1, y_2) \in U$ whenever $(y_1, y_2) \in X$ and $\rho_{sum}((x_1, x_2), (y_1, y_2)) < \delta$. For such $(x_1, x_2) \in U$, $\exists \delta_2 = \frac{\delta}{2} > 0$, if $\rho_{\max}((x_1, x_2), (y_1, y_2)) < \delta_2$, then $\rho_{sum}((x_1, x_2), (y_1, y_2)) \leq 2\rho_{\max}((x_1, x_2), (y_1, y_2)) < 2\delta_2 = \delta$, and hence $(y_1, y_2) \in U$. In other words, $\forall (x_1, x_2) \in U$, $\exists \delta_2 > 0$ such that $(y_1, y_2) \in U$ whenever $(y_1, y_2) \in X$ and $\rho_{\max}((x_1, x_2), (y_1, y_2)) < \delta_2$. This implies that $U \in \mathfrak{T}_{\rho_{\max}}$, and therefore $\mathfrak{T}_{\rho_{sum}} \subseteq \mathfrak{T}_{\rho_{\max}}$.
- For each $U \in \mathfrak{T}_{\rho_{\max}}$: $\forall (x_1, x_2) \in U$, $\exists \delta > 0$, such that $(y_1, y_2) \in U$ whenever $(y_1, y_2) \in X$ and $\rho_{\max}((x_1, x_2), (y_1, y_2)) < \delta$. For such $(x_1, x_2) \in U$, $\exists \delta_3 = \delta > 0$, if $\rho_E((x_1, x_2), (y_1, y_2)) < \delta_3$, then $\rho_{\max}((x_1, x_2), (y_1, y_2)) \leq \rho_E((x_1, x_2), (y_1, y_2)) < \delta_3 = \delta$, and hence $(y_1, y_2) \in U$. In other words, $\forall (x_1, x_2) \in U$, $\exists \delta_3 > 0$ such that $(y_1, y_2) \in U$ whenever $(y_1, y_2) \in X$ and $\rho_E((x_1, x_2), (y_1, y_2)) < \delta_3$. This implies that $U \in \mathfrak{T}_{\rho_E}$, and therefore $\mathfrak{T}_{\rho_{\max}} \subseteq \mathfrak{T}_{\rho_E}$.

Therefore, we have $\mathfrak{T}_{\rho_E} = \mathfrak{T}_{\rho_{sum}} = \mathfrak{T}_{\rho_{\max}}$.

(2) **Proof:** " $\Rightarrow$ " If $(\mathfrak{T}) - \lim\limits_{n\to\infty} x_n = x$, then for each $L \in \mathfrak{T}$ satisfying $x \in L$, there exists $N \in \mathbb{N}$ such that $x_n \in L$ whenever $n \geq N$. For any $\varepsilon > 0$, let $L_\varepsilon(x) = \{y \in X | \rho_j(y, x) < \varepsilon,\ j = E, \max \text{ or sum}\}$. $L_\varepsilon(x) \in \mathfrak{T}$ (In fact, for $\forall z \in L_\varepsilon(x)$, $\exists \delta = \dfrac{\varepsilon - \rho_j(z, x)}{2} > 0$ such that $w \in L_\varepsilon(x)$ whenever $w \in X$ and $\rho_j(w, z) < \delta$) and $x \in L_\varepsilon(x)$, so there exists $N \in \mathbb{N}$ such that $x_n \in L_\varepsilon(x)$(i.e. $\rho_j(x_n, x) < \varepsilon$) whenever $n \geq N$. This implies that $(\rho_j) - \lim\limits_{n\to\infty} x_n = x,\ j = E,\ \max$ or sum.

" $\Leftarrow$ " Conversely, suppose $(\rho_j) - \lim\limits_{n\to\infty} x_n = x$ ($j = E,\ \max$ or sum). For any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $\rho_j(x_n, x) < \varepsilon$ when $n \geq N$. So for each $L \in \mathfrak{T}$ satisfying $x \in L$, there exists $\delta > 0$ such that $\rho_j(x_n, x) < \delta$ for some $N \in \mathbb{N}$. This implies that $x_n \in L$ for $n \geq N$ with some $N \in \mathbb{N}$, and hence $(\mathfrak{T}) - \lim_{n\to\infty} x_n = x$.

(3) **Proof:** " $\Rightarrow$ " Suppose f is $(\mathfrak{T} - \mathfrak{R})$-continuous at $x \in X$. For any $\varepsilon > 0$, set $M_\varepsilon f(x) = \{y \in \mathbb{R} | \sigma(y, f(x)) < \varepsilon\}$, then $M_\varepsilon f(x) \in \mathfrak{R}$ (In fact, for each $z \in M_\varepsilon f(x)$, there is $\delta = \frac{\varepsilon - \sigma(f(x),z)}{2} > 0$ such that $w \in M_\varepsilon f(x)$ whenever $w \in \mathbb{R}$ and $\sigma(w, z) < \delta$) and $f(x) \in M_\varepsilon f(x)$. Since f is $(\mathfrak{T} - \mathfrak{R})-$ continuous at x, there is $L \in \mathfrak{T}$ such that $x \in L$ and $f(L) \subset M_\varepsilon f(x)$. By the definition of $\mathfrak{T}$, there is $\delta > 0$ such that $y \in L$ whenever $\rho_j(y, x) < \delta$. Since $f(L) \subset M_\varepsilon f(x)$, we have $\sigma(f(y), f(x)) < \varepsilon$ whenever $\rho_j(y, x) < \delta$. Therefore, f is $(\rho_j - \sigma)$-continuous at $x \in X$, where $j = E,\ \max$ or sum.

" $\Leftarrow$ " Conversely, suppose f is $(\rho_j - \sigma)$-continuous at $x \in X$, where $j = E,\ \max$ or sum. For any $\varepsilon > 0$, there exists $\delta > 0$ such that $\sigma(f(y), f(x)) < \varepsilon$ whenever $\rho_j(y, x) < \delta$. So for each $M \in \mathfrak{R}$ satisfying $f(x) \in M$, we have $\sigma(f(y), f(x)) < \varepsilon$ whenever $\rho_j(y, x) < \delta$. By the definition of $\mathfrak{R}$, we have $f(y) \in M$ whenever $\rho_j(y, x) < \delta$. Set $L_\delta(x) = \{y \in X | \rho_j(y, x) < \delta,\ j = E, \max \text{ or sum}\}$, then $L_\delta(x) \in \mathfrak{T}$, $x \in L_\delta(x)$ and $f(L_\delta(x)) \subset M$. Hence we have for each $M \in \mathfrak{R}$ satisfying $f(x) \in M$, there is $L_\delta(x) \in \mathfrak{T}$ such that $x \in L_\delta(x)$ and $f(L_\delta(x)) \subset M$. Therefore f is $(\mathfrak{T} - \mathfrak{R})$-continuous at $x \in X$.

(5) **Solution:** A subset of the metric space $(\mathbb{R}, \rho)$ is compact if and only if the subset has finite elements. If a subset A of metric space $(\mathbb{R}, \rho)$ has infinite many elements, then it is not totally bounded, since for $\varepsilon = 0.5$, $B_\varepsilon(x) = \{y \in \mathbb{R} : \rho(y, x) < \varepsilon\}$ covers only one element $x \in \mathbb{R}$, and we could not find finite $\{x_1, x_2, \cdots, x_n\}(n \in \mathbb{N})$ such that $A \subset \bigcup_{1\leq k\leq n} B_\varepsilon(x_k)$. So A is not compact.

If a subset A has finite elements, say $A = \{x_1, x_2, \cdots, x_n\}(n \in \mathbb{N})$, then it is totally bounded, since for any $\varepsilon > 0$, we have $A \subset \bigcup_{1\leq k\leq n} B_\varepsilon(x_k)$. Moreover, it is closed. Therefore, it is compact.