Example Let (X,ρ) be a non-empty metric space, (Y,σ) be a non-empty complete metric space. Let $C_b(X,Y)$ be the set of all continuous and bounded functions from X to Y,

i.e.

$$C_b(X,Y)=\{f \mid f: X\to Y \text{ is } \rho\text{-}\sigma \text{ continuous on } X, \text{ and } f(X) \text{ is bounded in } Y\}.$$

Define $\rho_s : C_b(X,Y)\times C_b(X,Y)\to \mathbb{R}$ by

$$\rho_s(f,g)=\sup\{\sigma(f(x),g(x)) : x\in X\}$$

(note that $\{\sigma(f(x),g(x)) : x\in X\}$ is bounded because $f(X)$, $g(X)$ are bounded and $\sigma(f(x),g(x))\le \sigma(f(x),p)+\sigma(p,q)+\sigma(q,g(x))$). It is easy to check that ρ_s is a metric on $C_b(X,Y)$. To see that $(C_b(X,Y),\rho_s)$ is complete, let $(f_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in $C_b(X,Y)$, and let $\varepsilon>0$. Then $\exists N\in\mathbb{N}$ such that

$$\rho_s(f_n,f_p)<\varepsilon \quad \text{whenever } n,p\ge N.$$

Thus (*) $\sigma(f_n(x),f_p(x))<\varepsilon$ whenever $n,p\ge N$, $x\in X$.

It follows that for each $x \in X$, $(f_n(x))_{n\in\mathbb{N}}$ is Cauchy in Y. As Y is complete, $(f_n(x))_{n\in\mathbb{N}}$ converges to an element, l_x say, in Y. Define $f: X \to Y$ by:

$$f(x) = l_x \ (= \lim_{n\to\infty} f_n(x)), \quad x \in X.$$

By $(\star)$, for all $x, y \in X$,

$$\sigma(f_n(x), f_p(x)) < \varepsilon, \quad \sigma(f_n(y), f_p(y)) < \varepsilon \quad \text{whenever } n, p \geq N.$$

Letting $p \to \infty$ we get (from an Exercise):

$$(\star') \quad \sigma(f_n(x), f(x)) \leq \varepsilon, \quad \sigma(f_n(y), f(y)) \leq \varepsilon \quad \text{whenever } n \geq N.$$

Because f_N is continuous at x, $\exists\, \delta > 0$ such that

$$(\star\star) \quad \sigma(f_N(x), f_N(y)) < \varepsilon \quad \text{whenever } y \in B_\delta(x).$$

By $(\star')$ & $(\star\star)$,

$$\sigma(f(x), f(y)) \leq \sigma(f(x), f_N(x)) + \sigma(f_N(x), f_N(y)) + \sigma(f_N(y), f(y))$$
$$< 3\varepsilon \quad \text{whenever } y \in B_\delta(x).$$

Therefore f is continuous at x, hence f is continuous on X. By $(\star')$, f is bounded (as so is f_N). Thus, $f \in C_b(X, Y)$.

By $(\star')$, $\rho_s(f_n, f) \leq \varepsilon$ whenever $n \geq N$. Thus $(f_n)_{n\in\mathbb{N}}$ converges to f in $C_b(X,Y)$. Therefore, $C_b(X,Y)$ is complete.

<u>Remark</u>

We construct an isometry $T: X \to C_b(X,\mathbb{R})$, where $(C_b(X,\mathbb{R}), \rho_s)$ is the $(C_b(X,Y), \rho_s)$ considered at the end of $\boxed{11}$ with Y replaced by $\mathbb{R}$ (for, the closure $\overline{T(X)}$ of $T(X)$ in the complete metric space $C_b(X,\mathbb{R})$ is a complete metric space). We may suppose $X \neq \phi$, and choose arbitrarily $x_0 \in X$.

For each $x \in X$, define $f_x : X \to \mathbb{R}$ by:

$$f_x(y) = \rho(x,y) - \rho(x_0,y).$$

It is easy to see that $f_x \in C_b(X,\mathbb{R})$, because:

(i) $|f_x(y)-f_x(y')| \leq |\rho(x,y)-\rho(x,y')| + |\rho(x_0,y')-\rho(x_0,y)|$

$$\leq 2\rho(y,y')$$

(hence f_x is continuous on X),

(ii) $|f_x(y)-f_{x'}(y)| = |\rho(x,y)-\rho(x',y)| \leq \rho(x,x')$,

in particular (as $f_{x_0}(y)=0 \ \forall y\in X$),

$$|f_x(y)| \leq \rho(x,x_0), \ \forall y\in X$$

(hence f_x is bounded).

Define $T: X \to C_b(X,\mathbb{R})$ by

$$T(x)=f_x \ , \quad x\in X.$$

Then by (ii) above,

$$\rho_s(T(x),T(x')) \leq \rho(x,x'), \text{ whenever } x,x'\in X.$$

Putting $y=x'$ into (ii), we get

$$|f_x(x')-f_{x'}(x')| = |\rho(x,x')| = \rho(x,x'),$$

so $\rho_s(T(x'),T(x)) = \sup\{|f_x(y)-f_{x'}(y)| : y\in X\}$

$\geq |f_x(x')-f_{x'}(x')| = \rho(x,x')$,

i.e. $\rho_s(T(x),T(x')) \geq \rho(x,x')$.

Therefore $\rho_s(T(x),T(x'))=\rho(x,x')$, T is an isometry, and we are done.