Example 6 Let $s = \{$all real sequences$\}$, and let $\rho : s \times s \to \mathbb{R}$ be defined by:

$$\rho(x,y) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{|\xi_n - \eta_n|}{1+|\xi_n - \eta_n|}, \quad x = (\xi_n),\ y = (\eta_n) \in s.$$

Then clearly

(i) for all $x, y \in s$, $\rho(x,y) \geq 0$, and $\rho(x,y) = 0$ iff $x = y$,

(ii) for all $x, y \in s$, $\rho(x,y) = \rho(y,x)$.

To see that

(iii) for all $x, y, z \in s$, $\rho(x,y) \leq \rho(x,z) + \rho(z,y)$,

it suffices to prove that for all real numbers a, b:

$$(*) \quad \frac{|a+b|}{1+|a+b|} \leq \frac{|a|}{1+|a|} + \frac{|b|}{1+|b|}.$$

The proof of $(*)$ breaks down into two parts.

(α) Suppose $ab \geq 0$ (i.e. a, b have the same sign). Then we can assume that $a, b \geq 0$. It follows that

$$\frac{|a+b|}{1+|a+b|} = \frac{a+b}{1+a+b} = \frac{a}{1+a+b} + \frac{b}{1+a+b} \leq \frac{|a|}{1+|a|} + \frac{|b|}{1+|b|}.$$

(β) Suppose $ab<0$. Then we can assume that $|a| \geq |b|$. It follows that $|a+b| \leq |a|$. Since $\frac{u}{1+u} - \frac{v}{1+v} = \frac{u-v}{(1+u)(1+v)} \leq 0$ if $0 \leq u \leq v$ in $\mathbb{R}$, we have

$$\frac{|a+b|}{1+|a+b|} \leq \frac{|a|}{1+|a|} \leq \frac{|a|}{1+|a|} + \frac{|b|}{1+|b|}.$$

Therefore, (*) holds, (iii) is true, and (s, ρ) is a metric space.

We now prove that for each sequence $(x_p)_{p \in \mathbb{N}} \subset s$, and each $z \in s$,

$$\rho\text{-}\lim_{p \to \infty} x_p = z \quad \text{iff} \quad \forall n \in \mathbb{N}, \ \lim_{p \to \infty} \xi_n^{(p)} = \eta_n \text{ in } \mathbb{R},$$

where $x_p = (\xi_n^{(p)})_{n \in \mathbb{N}}$ and $z = (\eta_n)_{n \in \mathbb{N}}$. Suppose that $\rho\text{-}\lim_{p \to \infty} x_p = z$, and let $n \in \mathbb{N}$. We want to show that $\lim_{p \to \infty} \xi_n^{(p)} = \eta_n$ in $\mathbb{R}$. To this end, let $\varepsilon > 0$. Define $\varepsilon' = \min\{\varepsilon, 1\}$. Because $\rho\text{-}\lim_{p \to \infty} x_p = z$,

$\exists\, N \in \mathbb{N}$ such that

$$\rho(x_p, z) < \varepsilon'/2^{n+1} \quad \text{whenever } p \geq N.$$

It follows that for all $p \geq N$,

$$\frac{1}{2^n}\,\frac{|\xi_n^{(p)} - \eta_n|}{1+|\xi_n^{(p)} - \eta_n|} < \frac{\varepsilon'}{2^{n+1}}\,,$$

i.e.
$$|\xi_n^{(p)} - \eta_n| < 2^{-1}\varepsilon'/(1-2^{-1}\varepsilon') \leq \varepsilon' \leq \varepsilon,$$

because $\varepsilon' \leq 1$, we have $1-2^{-1}\varepsilon' \geq 1-2^{-1} = 2^{-1}$, $2^{-1}\varepsilon'/(1-2^{-1}\varepsilon') \leq \varepsilon'$.

Hence,
$$|\xi_n^{(p)} - \eta_n| < \varepsilon \quad \text{whenever } p \geq N.$$

Thus $\displaystyle\lim_{p\to\infty} \xi_n^{(p)} = \eta_n$ for every $n \in \mathbb{N}$.

Conversely, suppose that $\forall n \in \mathbb{N}$, $\displaystyle\lim_{p\to\infty} \xi_n^{(p)} = \eta_n$ in $\mathbb{R}$. We want to show that $\rho\text{-}\displaystyle\lim_{p\to\infty} x_p = z$. To this end, let $\varepsilon > 0$. Since $\displaystyle\sum_{n=1}^{\infty} 2^{-n}$ converges to a finite number, $\exists\, M \in \mathbb{N}$ such that $\displaystyle\sum_{n=M+1}^{\infty} 2^{-n} < \varepsilon/2$. Hence

$$\sum_{n=M+1}^{\infty} \frac{1}{2^n}\,\frac{|\xi_n^{(p)} - \eta_n|}{1+|\xi_n^{(p)} - \eta_n|} \leq \sum_{n=M+1}^{\infty} 2^{-n} < \varepsilon/2\,.$$

For each $n = 1, 2, \cdots, M$, because $\lim_{p\to\infty} \xi_n^{(p)} = \eta_n$, $\exists\, P_n \in \mathbb{N}$ such that

$$|\xi_n^{(p)} - \eta_n| < \varepsilon/2 \quad \text{whenever } p \geq P_n.$$

Define $Q = \max\{P_n : n = 1, 2, \cdots, M\}$. Then for all $p \geq Q$,

$$\sum_{n=1}^{M} \frac{1}{2^n} \frac{|\xi_n^{(p)} - \eta_n|}{1 + |\xi_n^{(p)} - \eta_n|} \leq \sum_{n=1}^{M} \frac{1}{2^n} |\xi_n^{(p)} - \eta_n| \leq \left(\sum_{n=1}^{M} \frac{1}{2^n}\right) \varepsilon/2 < \varepsilon/2,$$

$$\sum_{n=1}^{\infty} \frac{1}{2^n} \frac{|\xi_n^{(p)} - \eta_n|}{1 + |\xi_n^{(p)} - \eta_n|} = \sum_{n=1}^{M} \frac{1}{2^n} \frac{|\xi_n^{(p)} - \eta_n|}{1 + |\xi_n^{(p)} - \eta_n|} + \sum_{n=M+1}^{\infty} \frac{1}{2^n} \frac{|\xi_n^{(p)} - \eta_n|}{1 + |\xi_n^{(p)} - \eta_n|}$$
$$< \varepsilon/2 + \varepsilon/2 = \varepsilon\,;$$

hence $\rho(x_p, z) < \varepsilon$ whenever $p \geq Q$. Thus, $\rho\text{-}\lim_{p\to\infty} x_p = z$.

With similar argument, we see that if $(x_p)_{p\in\mathbb{N}}$ is Cauchy in (s, ρ), and if $x_p = (\xi_n^{(p)})_{n\in\mathbb{N}}$, then $\forall\, n \in \mathbb{N}$, $(\xi_n^{(p)})_{p\in\mathbb{N}}$ is Cauchy in $\mathbb{R}$. Let $\varphi_n = \lim_{p\to\infty} \xi_n^{(p)}$, and let $w = (\varphi_n)_{n\in\mathbb{N}}$. Then, by the result of the last paragraph, $(x_p)_{p\in\mathbb{N}}$ converges to w in (s, ρ). Thus, (s, ρ) is complete.

(s, ρ) is not compact, because the sequence

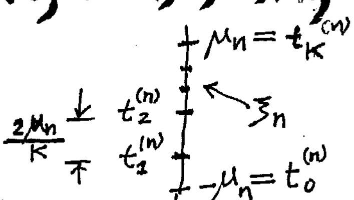
$(x_p)_{p\in\mathbb{N}}$, where $x_p = (p, p, \cdots)$ has no convergent subsequence in (s, ρ). We will prove that for every positive sequence $(\mu_n)_{n\in\mathbb{N}}$ with $\lim_{n\to\infty} \mu_n = 0$ (in $\mathbb{R}$), the subset $X \overset{\text{def}}{=} \{x = (\xi_n)_{n\in\mathbb{N}} \in s : |\xi_n| \le \mu_n \ \forall n\}$ of s is compact. Note that X is obviously closed.

To see that X is totally bounded, let $\varepsilon > 0$.

Let $M \in \mathbb{N}$ be such that $\sum_{n=M+1}^{\infty} 2^{-n} \le \varepsilon/2$, and

let $K \in \mathbb{N}$ be such that $K^{-1} 2 \operatorname{Max}\{\mu_n : n = 1, 2, \cdots, M\} \le \varepsilon/2$. Let

$$t_j^{(n)} = -\mu_n + \frac{j}{K} 2\mu_n, \quad j = 0, 1, \cdots, K, \ n = 1, 2, \cdots, M,$$



and let

$$x^{(j_1, j_2, \cdots, j_M)} = (t_{j_1}^{(1)}, t_{j_2}^{(2)}, \cdots, t_{j_M}^{(M)}, 0, 0, \cdots, 0, \cdots)$$

where $j_1, j_2, \cdots, j_M \in \{0, 1, \cdots, K\}$. Because $\forall x = (\xi_n)_{n\in\mathbb{N}} \in s$, $|\xi_n| \le \mu_n$ for all $n \in \mathbb{N}$ (in particular, for $n = 1, 2, \cdots, M$), it is easy to see that

$$(\dagger) \qquad X \subset \bigcup \{B_\varepsilon(x^{(j_1, j_2, \cdots, j_M)}) : j_1, j_2, \cdots, j_M = 0, 1, \cdots, \text{or } K\}.$$

Thus, X is compact.

To see $(\dagger)$, note that for each $x = (\xi_n) \in X$, we can choose j_n such that $|\xi_n - t_{j_n}^{(n)}| \le \frac{2\mu_n}{K}$ for $n = 1, 2, \cdots, M$. Then

$$\sum_{n=1}^{M} 2^{-n} \frac{|\xi_n - t_{j_n}^{(n)}|}{1 + |\xi_n - t_{j_n}^{(n)}|} < \sum_{n=1}^{M} 2^{-n} |\xi_n - t_{j_n}^{(n)}| \le \sum_{n=1}^{M} 2^{-n} \left(\frac{2\mu_n}{K}\right) \le \frac{\varepsilon}{2}.$$

Illustration

1. Let $\varepsilon = 1.5$.

Then $\rho(x, x^{(j_1,\cdots,j_M)}) \le \sum_{n=1}^{\infty} 2^{-n} = 1 < 1.5$,

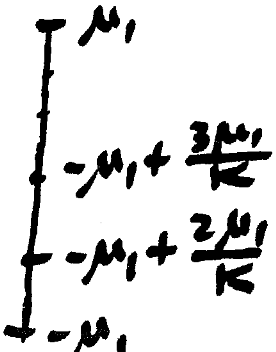
i.e. $\rho(x, x^{(j_1,\cdots,j_M)}) < \varepsilon$.

2. Let $\varepsilon = 1$.

Since $\sum_{n=2}^{\infty} 2^{-n} = \frac{1}{2}$, we take $M = 1$.

Pick $K \in \mathbb{N}$ such that $\frac{2\mu_1}{K} \le \frac{\varepsilon}{2} = \frac{1}{2}$

i.e. $K \ge 4\mu_1$.



Suppose $x \in X$ and $x = (\xi_1, \xi_2, \cdots)$. Then

$$\xi_1 \in [-\mu_1, \mu_1] = \bigcup_{j=1}^{K} \left[-\mu_1 + \frac{(j-1)2\mu_1}{K}, -\mu_1 + \frac{j2\mu_1}{K}\right].$$

Suppose

$$\xi_1 \in \left[-\mu_1 + \frac{2(2\mu_1)}{K},\ -\mu_1 + \frac{3(2\mu_1)}{K}\right].$$

Then, with $j_1 = 3$ and $x^{(j_1)} = (\underbrace{-\mu_1 + \frac{3(2\mu_1)}{K}}_{t_{j_1}^{(1)}}, 0, 0, \cdots)$,

we have

$$\rho(x, x^{(j_1)}) = \frac{1}{2}\,\frac{|\xi_1 - t_{j_1}^{(1)}|}{1 + |\xi_1 - t_{j_1}^{(1)}|} + \sum_{n=2}^{\infty} 2^{-n}\,\frac{|\xi_n - 0|}{1 + |\xi_n - 0|}$$

$$\le \frac{1}{2}\,|\xi_1 - t_{j_1}^{(1)}| + \sum_{n=2}^{\infty} 2^{-n}$$

$$\le 1\,\frac{2\mu_1}{K} + \frac{1}{2} \le \frac{1}{2} + \frac{1}{2} = 1,$$

i.e. $\rho(x, x^{(j_1)}) < \varepsilon$.