Even in a complete metric space, a bounded closed set need not be compact. For example, with $X = \mathbb{N}$, the (discrete) metric ρ considered in Example (2) on p.III.1 makes $\mathbb{N}$ a complete metric space, $\mathbb{N}$ itself is bounded and closed but non-compact (because the sequence $(n)_{n\in\mathbb{N}}$ does not have a convergent subsequence as $\rho(m,n) > \frac{1}{2}$ whenever $m \neq n$). To present a generalization of Heine-Borel Theorem, we need total boundedness. A subset A of a metric space (X,ρ) is <u>totally bounded</u> if for each $\varepsilon > 0$, $\exists \{x_1, x_2, \cdots, x_n\} \subset X$ ($n \in \mathbb{N}$) such that

$$A \subset \bigcup_{1\le k\le n} B_\varepsilon(x_k).$$

Clearly A is bounded if A is totally bounded, but the converse is false in general (consider $(\mathbb{N},\rho)$ above), though true for $\mathbb{R}^p$.

Thm. 28 Let (X, ρ) be a complete metric space, and $A \subset X$. Then A is compact iff A is closed and totally bounded.

Pf. ($\Rightarrow$) Suppose A is compact. By Prop. 19, A is closed. To see that A is totally bounded, we suppose the contrary and construct a sequence in A which does not have a convergent subsequence. Thus, $\exists\, \varepsilon > 0$ such that for each $n \in \mathbb{N}$ and for every finite subset $\{z_1, z_2, \cdots, z_n\} \subset X$, $A \not\subset \bigcup_{1 \le k \le n} B_\varepsilon(z_k)$.
So $A \neq \phi$, and we may choose arbitrarily $x_1 \in A$.
As $A \not\subset B_\varepsilon(x_1)$, $A \setminus B_\varepsilon(x_1) \neq \phi$ and we choose $x_2 \in A \setminus B_\varepsilon(x_1)$. Then $\rho(x_1, x_2) \geq \varepsilon$. Inductively, for an integer $n \geq 3$, we pick $x_n \in A \setminus \bigcup_{k=1}^{n-1} B_\varepsilon(x_k)$.
Then $\rho(x_n, x_k) \geq \varepsilon$ for all $k = 1, 2, \cdots, n-1$. Obviously the sequence (x_n) is in A, but (x_n) does not have a convergent subsequence. This is impossible because A is compact. Therefore A is totally bounded.

($\Leftarrow$) Suppose A is totally bounded and closed. To see that A is compact, let $(a_n)_{n\in\mathbb{N}}$ be a sequence in A, and we will show that (a_n) has a Cauchy subsequence (then by the completeness of X and the closedness of A, this subsequence converges to an element of A). Denote $A_0 = A$, $n_0 = 1$. Because A_0 is totally bounded, $\exists \{q_1, q_2, \cdots, q_m\} \subset X$ $(m \in \mathbb{N})$ such that $A_0 \subset \bigcup_{k=1}^{m} B_1(q_k)$. Thus $\exists p_1 \in \{q_1, q_2, \cdots, q_m\}$ such that $\{n \in \mathbb{N} : a_n \in A_0 \cap B_1(p_1)\}$ is infinite. Define

$$A_1 = A_0 \cap B_1(p_1)$$

and choose $n_1 \in \mathbb{N}$ such that $a_{n_1} \in A_1$ and $n_1 > n_0$.

Being a subset of A_0 (which is totally bounded), A_1 is also totally bounded. By an argument similar to the above, $\exists p_2 \in X$ such that $\{n \in \mathbb{N} : a_n \in A_1 \cap B_{1/2}(p_2)\}$ is infinite. Define

$$A_2 = A_1 \cap B_{1/2}(p_2)$$

and choose $n_2 \in \mathbb{N}$ such that $a_{n_2} \in A_2$ and $n_2 > n_1$. Proceed inductively, for an integer $k \geq 3$, $\exists\, p_k \in X$ such that $\{n \in \mathbb{N} : a_n \in A_{k-1} \cap B_{1/k}(p_k)\}$ is infinite; define

$$A_k = A_{k-1} \cap B_{1/k}(p_k)$$

and choose $n_k \in \mathbb{N}$ such that $a_{n_k} \in A_k$ and $n_k > n_{k-1}$.

Then $\forall\, k \in \mathbb{N}$, $a_{n_j} \in A_j \subset A_k = A_{k-1} \cap B_{1/k}(p_k)$ whenever $j \geq k$; thus for all $j, j' \geq k$,

$$\begin{aligned}\rho(a_{n_j}, a_{n_{j'}}) &\leq \rho(a_{n_j}, p_k) + \rho(p_k, a_{n_{j'}}) \\ &< 1/k + 1/k = 2/k.\end{aligned}$$

It follows that $(a_{n_j})_{j \in \mathbb{N}}$ is Cauchy, and we are done.

Cor. A metric space is compact iff it is complete and totally bounded.

13

$\boxed{13}$ Let (X, ρ) be a metric space. A contraction of X is a mapping $f : X \to X$ such that for some constant $k \in (0,1)$,

$$(*) \qquad \rho(f(x), f(y)) \leq k\, \rho(x, y)$$

for all $x, y \in X$. Note that in general, we cannot replace (*) by

$$\rho(f(x), f(y)) < \rho(x,y) \text{ whenever } x \neq y \text{ in } X.$$

For example, let $X = (0, \frac{1}{2})$ with the usual metric, and $f: X \to X: x \mapsto x^2$. Then $|f(x) - f(y)| < |x-y|$ whenever $x \neq y$ in X, but there does not exist $k \in (0,1)$ such that (*) is satisfied. Moreover, there does not exist $x \in X$ such that $f(x) = x$.

Thm. 29 Let (X, ρ) be a complete metric space, and $f: X \to X$ be a contraction. Then

(i) for every $x \in X$, $f^{(n)}(x) = \underbrace{f \circ f \circ \cdots \circ f}_{n \text{ of them}}(x)$ converges to an element p in X (as $n \to \infty$),

(ii) $f(p) = p$,

(iii) p is unique (independent of x).

Pf. Denote $x_0 = x$. For each $n \in \mathbb{N}$, define

$$x_n = f^{(n)}(x).$$

Then $x_n = f(x_{n-1})$,

$$\begin{aligned}\rho(x_n, x_{n+1}) &= \rho(f(x_{n-1}), f(x_n)) \leq k \rho(x_{n-1}, x_n) \quad (\text{by}(*)) \\ &\leq k^2 \rho(x_{n-2}, x_{n-1}) \leq \cdots \\ &\leq k^n \rho(x_0, x_1).\end{aligned}$$

We will see that $(x_n)_{n\in\mathbb{N}}$ is Cauchy. Indeed, let $\varepsilon>0$. Choose $N\in\mathbb{N}$ such that

$$\frac{k^N}{1-k}\rho(x_0,x_1)<\varepsilon .$$

Then for all $m,n\in\mathbb{N}$ satisfying $n\geq m\geq N$,

$$\rho(x_m,x_n)\leq\sum_{j=0}^{n-m-1}\rho(x_{m+j},x_{m+j+1})$$

$$\leq\sum_{j=0}^{n-m-1}k^{m+j}\rho(x_0,x_1)=k^m\rho(x_0,x_1)\sum_{j=0}^{n-m-1}k^j$$

$$\leq k^N\frac{1}{1-k}\rho(x_0,x_1)<\varepsilon .$$

Thus (x_n) is Cauchy. Because (X,ρ) is complete, (x_n) converges to some element, say p, in X. Because f is continuous, $x_{n+1}=f(x_n)\to f(p)$ (as $n\to\infty$). Because $\lim_{n\to\infty}x_{n+1}=\lim_{n\to\infty}x_n=p$, $p=f(p)$. Such a p is unique, because if $q=f(q)$, $q\in X$, then

$$\rho(p,q)=\rho(f(p),f(q))\leq k\rho(p,q),$$

$$(1-k)\rho(p,q)\leq 0,$$

$$\rho(p,q)=0 \text{ (as } k\in(0,1)) \text{ and } p=q .$$

Thm 11 (Picard's theorem)

Suppose that $F: \mathbb{R} \to \mathbb{R}$ is continuous and that for some real positive constant number L,

$$|F(y) - F(y')| \le L |y - y'|$$

for all $y, y' \in \mathbb{R}$. Let $y_0 \in \mathbb{R}$. Then there exist $\tau \in (0, \infty)$ and a unique function $y : [-\tau, \tau] \to \mathbb{R}$ such that y is differentiable on $[-\tau, \tau]$ and satisfies

$$\begin{cases} y'(x) = F(y(x)), & x \in [-\tau, \tau], \\ y(0) = y_0 \end{cases}$$

Pf. Arbitrarily choose $\tau \in (0, L^{-1})$. Define $T: C_b([-\tau, \tau], \mathbb{R}) \to C_b([-\tau, \tau], \mathbb{R})$: $f \mapsto T(f)$, where

$$T(f)(x) = y_0 + \int_0^x F(f(s))\, ds, \quad x \in [-\tau, \tau].$$

Note that indeed $T(f) \in C_b([-\tau, \tau], \mathbb{R})$ (e.g. by continuity of $F \circ f$ and the compactness of $[-\tau,\tau]$, $F[f([-\tau,\tau])]$ is bounded).

Now, for all $f, g \in C_b([-\tau,\tau], \mathbb{R})$:

$$\|T(f)-T(g)\| = \sup\left\{ \left| \int_0^x [F(f(s))-F(g(s))]\, ds \right| : x \in [-\tau,\tau] \right\}$$

$$\leq \sup\left\{ \left| \int_0^x |F(f(s))-F(g(s))|\, ds \right| : x \in [-\tau,\tau] \right\}$$

$$\leq \sup\left\{ L \left| \int_0^x |f(s)-g(s)|\, ds \right| : x \in [-\tau,\tau] \right\}$$

$$\leq L\, \|f-g\| \sup\{ |x| : x \in [-\tau,\tau] \}$$

$$= \tau L\, \|f-g\|.$$

Because $\tau L < 1$, T is a contraction on the Banach space $C_b([-\tau,\tau];\mathbb{R})$. By Banach's fixed point theorem, there exists uniquely a fixed point f of T in $C_b([-\tau,\tau];\mathbb{R})$, i.e. $\exists! f \in C_b([-\tau,\tau];\mathbb{R}) = C([-\tau,\tau])$ such that

$$f = T(f),$$

or

$$f(x) = y_0 + \int_0^x F(f(s))\, ds, \quad x \in [-\tau,\tau]$$

which implies that f is differentiable on $[-\tau,\tau]$ and

$$f'(x) = F(f(x)), \quad x \in [-\tau,\tau],$$

and

$$f(0) = y_0 .$$

This proves the theorem. ■

IV.8

VII.13