$\boxed{12}$ Let (X, ρ) be a metric space, and $S \subset X$. S is said to be <u>compact</u> if every sequence $(a_n)_{n \in \mathbb{N}}$ in S has a subsequence $(a_{n_k})_{k \in \mathbb{N}}$ which converges to some $l \in S$. S is said to be <u>bounded</u> if $\exists\, p \in X, \exists\, r > 0$ such that $S \subset B_r(p)$, or equivalently, $\exists\, t > 0$ such that for all $x, y \in S$, $\rho(x, y) \leq t$.

<u>Prop.19</u> If S is compact, then S is closed and bounded.

<u>Prop.20</u> If S is compact, and $C \subset S$ is closed (w.r.t. ρ_S), then C is compact (w.r.t. ρ_S or ρ).

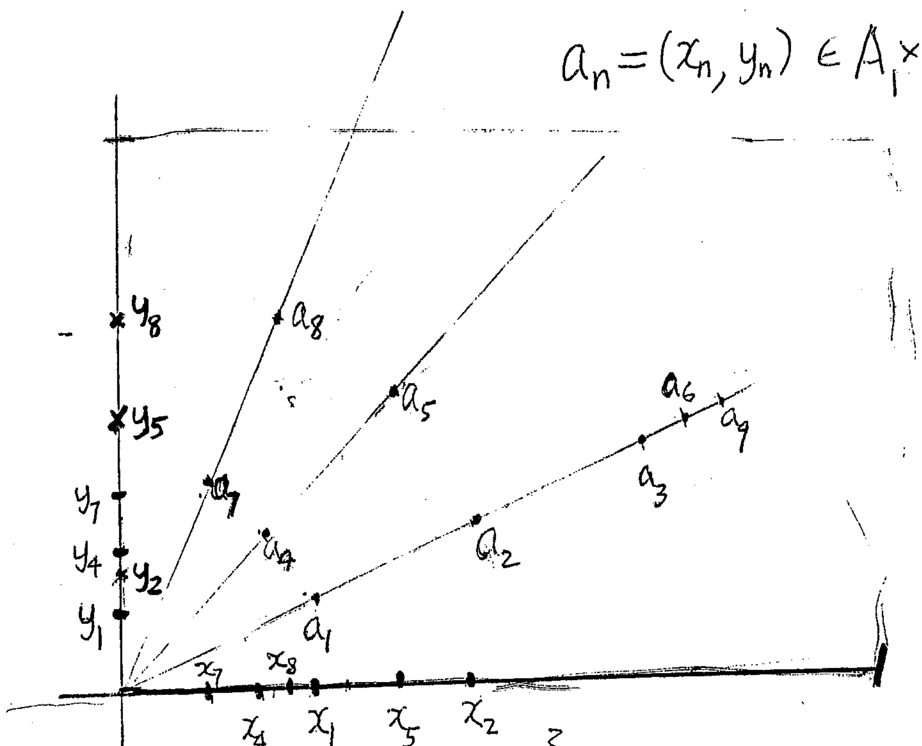
<u>Thm.21</u> Let (X, ρ_E) be the product metric space of the metric spaces (X_j, ρ_j), $j = 1, 2, \cdots, n$. Let A_j be a compact subset of X_j, $j = 1, 2, \cdots, n$. Then $A_1 \times A_2 \times \cdots \times A_n$ is compact (w.r.t. ρ_E).

(P. III. 8) Prop. 19

Pf. Suppose S is not bounded. Then $S \neq \phi$. Let $p \in S$. Because S is not bounded, we may let $a_n \in S \setminus B_n(p)$. Then $\rho(a_n, p) \geq n$. The sequence (a_n) does not have a convergent subsequence $(a_{n_k})_{k \in \mathbb{N}}$ because $\rho(a_{n_k}, p) \geq n_k \longrightarrow \infty$ as $k \to \infty$, while $(a_{n_k})_{k \in \mathbb{N}}$ (being convergent) is bounded. Therefore S is not compact. We conclude: if S is compact, then S is bounded.

$a_n = (x_n, y_n) \in A_1 \times A_2$

Thm. 21



Consider the sequence $(x_n)_{n \in \mathbb{N}}$ in X_1. Since X_1 is compact, there is a subsequence of (x_n) which converges to some element of X_1, ξ say. To avoid tedious notation, let us denote this subsequence as (x_n) also. Now, (y_n) is in X_2 which is also compact. So, $\exists\, n_1 < n_2 < \cdots < n_k < n_{k+1} < \cdots$ in $\mathbb{N}$ such that $(y_{n_k})_{k \in \mathbb{N}}$ converges to some element of X_2, η say.

$x_1, x_2, x_4, x_5, x_7, x_8, \ldots \longrightarrow 0 = \xi \in X_1$

$\in X_2$

$y_1, y_4, \quad y_7, \cdots \to \eta$ (say)

$a_1, a_4, \quad a_7, \cdots \to (\xi, \eta) \in X_1 \times X_2$

Then, as $(x_{n_k})_{k \in \mathbb{N}}$ converges to ξ also, we have $a_{n_k} = (x_{n_k}, y_{n_k}) \longrightarrow (\xi, \eta) \in X_1 \times X_2$ as $k \to \infty$. This completes the proof.

Thm. 22 (Heine-Borel)

Let $A \subset \mathbb{R}^p$. Then A is compact iff A is closed and bounded.

Thm. 23 (Cantor)

If $A_1 \supset A_2 \supset \cdots$ is a sequence of non-empty compact subsets of a metric space X, then $\bigcap_{n \in \mathbb{N}} A_n$ is non-empty and compact.

Thm. 24 Let (X, ρ), (Y, σ) be metric spaces, $f : X \to Y$ be continuous (w.r.t. ρ-σ), and let C be a compact subset of X. Then $f(C)$ is compact (w.r.t. σ).

Cor. Let $f : C \to \mathbb{R}$ be continuous, where C is a compact subset of $\mathbb{R}^p$ and $C \neq \phi$. Then there are $a, b \in C$ such that for all $x \in C$,

$$f(a) \leq f(x) \leq f(b).$$

Thm 24

not for tests?

Pf. Let (y_n) be a sequence in $f(C)$, i.e. $y_n \in f(C)$ for each $n \in \mathbb{N}$. Let $x_n \in C$ be such that $y_n = f(x_n)$. Because C is compact, $\exists$ a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ of (x_n) such that $\lim_{k\to\infty} x_{n_k} = x$ for some $x \in C$. By continuity of f at x, $f(x) = \lim_{k\to\infty} f(x_{n_k}) = \lim_{k\to\infty} y_{n_k}$.

Thus $(y_{n_k})_{k\in\mathbb{N}}$ converges to $f(x) \in f(C)$. Therefore $f(C)$ is compact.

Cor

Pf Because $f(C)$ is compact, and because $\inf f(C)$ and $\sup f(C)$ are limit points of $f(C)$ (check this!), $\inf f(C), \sup f(C) \in f(C)$.

Thus, $\exists\, a, b \in C$ such that $f(a) = \inf f(C)$, $f(b) = \sup f(C)$.

$\inf f(C)$, $\sup f(C)$ are limit points of $f(C)$

To see this, it suffices to prove that for any bounded below non-empty subset S of $\mathbb{R}$, $\inf S$ is a limit point of S. Let $\alpha = \inf S$. Then $\forall k \in \mathbb{N}$, $\exists\, s_k \in S$ such that $s_k < \alpha + \frac{1}{k}$ (recall (S) on p. I.5); but we also have $\alpha \le s_k$ (because $s_k \in S$ and α is a lower bound of S). By Sandwich thm, $\lim_{k\to\infty} s_k = \alpha$. Hence α is a limit point of S.

Thm 25 Let (X,ρ), (Y,σ) be metric spaces, $f: X \to Y$ be continuous and bijective, and X be compact. Then f^{-1} is continuous, i.e. f is a homeomorphism.

Proof. Denote $g = f^{-1}$. It suffices to show that $g^{-1}(B)$ is closed for every closed subset B of X. Now, B is compact by Prop. 20, and $g^{-1}(B) = f(B)$ is compact (by Thm. 24) hence closed by Prop. 19.

<u>Thm. 26</u> Let (X, ρ), (Y, σ) be metric space, X be compact, and $f: X \to Y$ be $(\rho\text{-}\sigma)$ continuous. Then f is uniformly continuous on X in the following sense:

$\forall \varepsilon > 0 \ \exists \delta > 0$ such that

$\sigma(f(x), f(x')) < \varepsilon$ whenever $x, x' \in X$ and $\rho(x, x') < \varepsilon$.

<u>thm. 26</u>

<u>Pf.</u> Suppose $\exists \varepsilon_1 > 0, \forall \delta > 0, \exists x', x \in X$ satisfying $\rho(x', x) < \delta$, $\sigma(f(x), f(x')) \geq \varepsilon_1$.

Then for each $n \in \mathbb{N}$, $\exists x_n', x_n \in X$, such that $\rho(x_n', x_n) < \frac{1}{n}$, $\sigma(f(x_n), f(x_n')) \geq \varepsilon_1$.

Because X is compact, $\exists$ a subsequence $(x'_{n_k})_{k \in \mathbb{N}}$ such that $\lim_{k\to\infty} x'_{n_k} = a$ for some $a \in X$.

It follows that $\lim_{k\to\infty} x_{n_k} = a$ (by $\rho(x_{n_k}, a) \leq \rho(x_{n_k}, x'_{n_k}) + \rho(x'_{n_k}, a) < \frac{1}{n_k} + \rho(x'_{n_k}, a) \to 0$ as $k \to \infty$) and (by continuity of f at a) $\lim_{k\to\infty} f(x_{n_k}) = f(a) = \lim_{k\to\infty} f(x'_{n_k})$, $\lim_{k\to\infty} \sigma(f(x_{n_k}), f(x'_{n_k})) = 0$. However, we have $\sigma(f(x_{n_k}), f(x'_{n_k})) \geq \varepsilon_1 > 0$. This contradiction shows that f is uniformly continuous on X.

Examples

1. $f : (0,\infty) \to \mathbb{R} : x \mapsto 1/x$ is continuous on $(0,\infty)$, but not uniformly continuous on $(0,\infty)$.

2. Let $g : (0,\infty) \to \mathbb{R} : x \mapsto x \sin\frac{1}{x}$. Then g is uniformly continuous on $(0,\infty)$.

3. Let (X,ρ) be the metric space considered in (2), p. III.1. What can you say about (a_n) in X, if (a_n) is convergent?

4. Let $S = (0,1)$, $p=1$ in Example (3), p. III.2. Let $f_n(x) = \frac{1}{nx}$, $x \in (0,1)$, $n \in \mathbb{N}$, $f(x) = 0$, $x \in (0,1)$. Determine whether

(a) $\lim_{n\to\infty} f_n(x) = f(x)$, $x \in (0,1)$?

(b) $f_n \to f$ in (X,ρ), i.e. $\lim_{n\to\infty} f_n = f$ w.r.t. ρ?

Ad(1) f is continuous on $(0,\infty)$, because for each $x\in(0,\infty)$ and for each $\varepsilon>0$, with $\delta \stackrel{\text{def}}{=} \min\left(\frac{x}{2}, \frac{x^2\varepsilon}{2}\right)$, we have

$$|f(x)-f(y)| = \left|\frac{1}{x}-\frac{1}{y}\right| = \frac{|x-y|}{xy} < \frac{\delta}{x}\cdot\frac{2}{x} \leq \varepsilon,$$

whenever $y\in(0,\infty)$ and $|y-x|<\delta$ (which implies $y\geq x-|x-y| > x-\delta \geqslant \frac{x}{2}$).

f is not uniformly continuous on $(0,\infty)$, because there is $\varepsilon_1>0$ such that for each $\delta>0$ we have

$$|f(x_1)-f(y_1)|\geq \varepsilon_1 \qquad (\dagger)$$

for some $x_1, y_1\in(0,\infty)$ satisfying $|x_1-y_1|<\delta$. To see this, let $\delta>0$ be given, let $\delta_1=\min(\delta,1)$, $x_1=\frac{\delta_1}{2}$, and $y_1=\frac{\delta_1}{3}$.

Then $|x_1-y_1|=\frac{\delta_1}{6}<\delta$, and

$$|f(x_1)-f(y_1)| = \frac{3}{\delta_1}-\frac{2}{\delta_1} = \frac{1}{\delta_1}\geq 1.$$

Ad(2) We want to show that $\forall\varepsilon>0, \exists\delta>0$ such that

(•) $|g(x)-g(y)|<\varepsilon$ whenever $x,y\in(0,\infty)$ and $|x-y|<\delta$,

where $g(t)=t\sin\frac{1}{t}$, $t\in(0,\infty)$. To this end, note that

(i) for each $\varepsilon>0$, with $\delta_1 \stackrel{\text{def}}{=} \varepsilon/2$, we have $\delta_1>0$ and

(*) $|g(x)|<\varepsilon/2$ whenever $x\in(0,\infty)$ and $x<\delta_1$;

(ii) for all $x, y \geq \delta_1/2$,

$$(**) \qquad |g(x)-g(y)| \leq \left(1+\frac{2}{\delta_1}\right)|x-y|,$$

because $|\sin\theta - \sin\phi| = \left|2\sin\frac{\theta-\phi}{2}\cos\frac{\theta+\phi}{2}\right| \leq 2\left|\sin\frac{\theta-\phi}{2}\right| \leq |\theta-\phi|$ (as $|\sin t| \leq |t|$ $\forall t \in \mathbb{R}$), $|g(x)-g(y)| \leq \left|(x-y)\sin\frac{1}{x}\right| + |y|\left|\sin\frac{1}{x} - \sin\frac{1}{y}\right|$ $\leq |x-y| + |y|\left|\frac{1}{x}-\frac{1}{y}\right| = \left(1+\frac{1}{x}\right)|x-y| \leq \left(1+\frac{2}{\delta_1}\right)|x-y|$.

Let $\delta \overset{\text{def}}{=} \min\left(\frac{\delta_1}{2}, \frac{\varepsilon}{1+2/\delta_1}\right) = \min\left(\frac{\varepsilon}{4}, \frac{\varepsilon^2}{\varepsilon+4}\right)$, and let $x, y > 0$ be such that $|x-y| < \delta$. If $x < \frac{\delta_1}{2}$, then $y < \delta_1$ (as $y \leq |y-x| + x < \delta + \frac{\delta_1}{2} \leq \delta_1$), so by $(*)$:

$$|g(x)-g(y)| \leq |g(x)| + |g(y)| < \varepsilon.$$

Similarly, if $y < \frac{\delta_1}{2}$, then $x < \delta_1$ and $|g(x)-g(y)| < \varepsilon$ by $(*)$.

Suppose therefore $x, y \geq \delta_1/2$. Then by $(**)$:

$$|g(x)-g(y)| \leq (1+2/\delta_1)|x-y| < (1+2/\delta_1)\delta \leq \varepsilon.$$

Thus $(\cdot)$ is proved, i.e. g is uniformly continuous on $(0,\infty)$.

<u>Thm. 27</u> Let (X,ρ) be a metric space, and $S \subset X$ be compact. Then (S, ρ_S) is complete.

Pf of Thm.27 To prove that (S, ρ_S) is complete, let $(a_n) \subset S$ be Cauchy. Then, as S is compact, there exist a subsequence $(a_{n_k})_{k \in \mathbb{N}}$ of (a_n), and an $\ell \in S$, such that $\ell = \lim_{k \to \infty} a_{n_k}$ w.r.t. ρ_S. Since (a_n) is Cauchy, we have $\ell = \lim_{n \to \infty} a_n$ too (by (2) of p. I.11, p. I.12). Therefore, (S, ρ_S) is complete.