Examples

1. Let (X,ρ) be a metric space. Then $\rho: X\times X \to \mathbb{R}$ is continuous w.r.t. ρ_E (or ρ_{max} or ρ_{sum}) and the usual metric on $\mathbb{R}$.

2. The addition and the scalar multiplication from $\mathbb{R}^p \times \mathbb{R}^p$, $\mathbb{R}\times\mathbb{R}^p$ respectively, to $\mathbb{R}^p$ are continuous. Consequently, for each $x \in \mathbb{R}^p$ and $\lambda \in \mathbb{R}\setminus\{0\}$, the mappings $t_x: \mathbb{R}^p \to \mathbb{R}^p : y \mapsto y+x$, $m_\lambda : \mathbb{R}^p \to \mathbb{R}^p : y \mapsto \lambda\cdot y$ are linear homeomorphisms.

V.1

Supplementary explanation

p. III.2

Example (3)

To prove the triangle inequality for ρ, let $f, g, h \in X$ and let $s \in S$. Then

$$|f(s)-g(s)| \leq |f(s)-h(s)| + |h(s)-g(s)| \quad \left(\begin{array}{c}\text{triangle inequality}\\ \text{for real numbers}\end{array}\right)$$

$$\leq \sup\{|f(t)-h(t)| : t \in S\} + \sup\{|h(t)-g(t)| : t \in S\},$$

i.e. $|f(s)-g(s)| \leq \rho(f,h) + \rho(h,g), \ \forall s \in S.$

$\therefore \sup\{|f(s)-g(s)| : s \in S\} \leq \rho(f,h) + \rho(h,g),$

i.e. $\rho(f,g) \leq \rho(f,h) + \rho(h,g).$

Example (5)

To prove the triangle inequality for ρ_E, denote $x = (x_1, x_2)$, $y = (y_1, y_2)$, and $z = (z_1, z_2)$ where $x_i, y_i, z_i \in X_i$ for $i = 1, 2$, and observe that (by the Cauchy Schwartz inequality for real numbers):

$$\sum_{i=1}^{2} \rho_i(x_i, z_i)\,\rho_i(z_i, y_i) \leq \left[\sum_{i=1}^{2} \rho_i(x_i, z_i)^2\right]^{1/2} \left[\sum_{i=1}^{2} \rho_i(z_i, y_i)^2\right]^{1/2}$$

i.e. (†) $\displaystyle\sum_{i=1}^{2} \rho_i(x_i,z_i)\rho_i(z_i,y_i) \leq \rho_E(x,z) \cdot \rho_E(z,y)$ (by def. of ρ_E).

Because

$$\rho_E(x,y)^2 = \rho_1(x_1,y_1)^2 + \rho_2(x_2,y_2)^2$$

$$\leq [\rho_1(x_1,z_1) + \rho_1(z_1,y_1)]^2 + [\rho_2(x_2,z_2) + \rho_2(z_2,y_2)]^2$$

$$= \left[\rho_1(x_1,z_1)^2 + \rho_2(x_2,z_2)^2\right] + \left[\rho_1(z_1,y_1)^2 + \rho_2(z_2,y_2)^2\right]$$
$$+ 2\left[\sum_{i=1}^{2} \rho_i(x_i,z_i)\rho_i(z_i,y_i)\right]$$
$$\leq \rho_E(x,z)^2 + \rho_E(z,y)^2 + 2\,\rho_E(x,z)\cdot\rho_E(z,y)$$
$$\therefore \quad \rho_E(x,y)^2 \leq \left[\rho_E(x,z) + \rho_E(z,y)\right]^2$$
$$\therefore \quad \rho_E(x,y) \leq \rho_E(x,z) + \rho_E(z,y).$$

p. III.4

Thm. 10

Ad(2) To prove that $\rho_E\text{-}\lim_{n\to\infty} a_n = l \; (= (l_1, l_2))$ iff $\rho_{max}\text{-}\lim_{n\to\infty} a_n = l$, suppose first that $\rho_E\text{-}\lim_{n\to\infty} a_n = l$. Let $\varepsilon>0$. Then $\exists N\in\mathbb{N}$ such that $\rho_E(a_n, l) < \varepsilon$ whenever $n \geq N$. Because by (1) of Thm. 10 (the present thm, p. III.4), $\rho_{max}(a_n, l) \leq \rho_E(a_n, l)$, we obtain:

$$\rho_{max}(a_n, l) < \varepsilon \quad \text{whenever } n \geq N.$$

Therefore $\rho_{max}\text{-}\lim_{n\to\infty} a_n = l$. Conversely, suppose that $\rho_{max}\text{-}\lim_{n\to\infty} a_n = l$. Let $\varepsilon>0$. Then $\exists M\in\mathbb{N}$ such that $\rho_{max}(a_n, l) < \varepsilon$ whenever $n \geq M$. By (1) of the present theorem, $\rho_E(a_n, l) \leq 2\,\rho_{max}(a_n, l)$, we have:

$$\rho_E(a_n, l) < 2\varepsilon \quad \text{whenever } n \geq M.$$

Hence $\rho_E\text{-}\lim_{n\to\infty} a_n = l$.

To prove that $\rho_{max}\text{-}\lim_{n\to\infty} a_n = l$ iff $\rho_j\text{-}\lim_{n\to\infty} a_{n,j} = l_j$, $j=1,2$, where $a_n=(a_{n,1}, a_{n,2})$, $l=(l_1,l_2)$, assume first that $\rho_{max}\text{-}\lim_{n\to\infty} a_n = l$. Let $\varepsilon>0$. Then $\exists N\in\mathbb{N}$ such that $\rho_{max}(a_n, l)<\varepsilon$ whenever $n\geq N$,

i.e. $\max\{\rho_1(a_{n,1}, l_1), \rho_2(a_{n,2}, l_2)\}<\varepsilon$ whenever $n\geq N$.

Thus $\rho_1(a_{n,1}, l_1)<\varepsilon$, $\rho_2(a_{n,2}, l_2)<\varepsilon$ whenever $n\geq N$.

Therefore $\rho_j\text{-}\lim_{n\to\infty} a_{n,j} = l_j$, $j=1,2$. Conversely, suppose $\rho_j\text{-}\lim_{n\to\infty} a_{n,j} = l_j$, $j=1,2$. Let $\varepsilon>0$. Then for $j=1,2$, $\exists M_j\in\mathbb{N}$ such that $\rho_j(a_{n,j}, l_j)<\varepsilon$ whenever $n\geq M_j$. Let $M=\max\{M_j : j=1,2\}$. Then $\forall\, n\geq M$, $n\geq M_j$ for $j=1,2$, hence $\rho_j(a_{n,j}, l_j)<\varepsilon$ for $j=1,2$. Therefore,

$$\rho_{max}(a_n, l)<\varepsilon \quad \text{whenever } n\geq M.$$

Thus $\rho_{max}\text{-}\lim_{n\to\infty} a_n = l$.

The other equivalences are similarly proved.

Pf of the Corollary Suppose X is complete w.r.t. ρ_{sum}. To prove that X_1 is complete w.r.t. ρ_1, let $(x_n^{(1)})_{n\in\mathbb{N}}$ be Cauchy in (X_1, ρ_1). Let $x^{(2)}\in X_2$ be arbitrarily fixed. Then, denoting $x_n = (x_n^{(1)}, x^{(2)})$, $n\in\mathbb{N}$, as we have $\rho_{sum}(x_n, x_m) = \rho_1(x_n^{(1)}, x_m^{(1)}) + \rho_2(x^{(2)}, x^{(2)})$, we see that $(x_n)_{n\in\mathbb{N}}$ is Cauchy in (X, ρ_{sum}). Since (X, ρ_{sum}) is complete, $\exists\, y=(y^{(1)}, y^{(2)})\in X$ such that $\rho_{sum}\text{-}\lim_{n\to\infty} x_n = y$. By (2) of Thm.10 on p.III.4, $\rho_1\text{-}\lim_{n\to\infty} x_n^{(1)} = y^{(1)}$. Therefore (X_1, ρ_1) is complete. The case for (X_2, ρ_2) is similar.

Conversely, suppose (X_j, ρ_j) is complete for $j=1,2$. To prove that (X, ρ_{sum}) is complete, let $x_n = (x_n^{(1)}, x_n^{(2)})$, $n \in \mathbb{N}$, be a Cauchy sequence in (X, ρ_{sum}). Then for each $\varepsilon > 0$, $\exists N \in \mathbb{N}$ such that

$$\rho_{sum}(x_n, x_m) < \varepsilon \qquad \text{whenever } n, m \geq N.$$

Thus $\rho_1(x_n^{(1)}, x_m^{(1)}) + \rho_2(x_n^{(2)}, x_m^{(2)}) < \varepsilon$ whenever $n, m \geq N$ (by def. of ρ_{sum}). Hence $\rho_j(x_n^{(j)}, x_m^{(j)}) < \varepsilon$ whenever $n, m \geq N$, $j=1,2$. Thus $(x_n^{(j)})_{n\in\mathbb{N}}$ is Cauchy in (X_j, ρ_j). Since (X_j, ρ_j) is complete for $j=1,2$, $\exists\, y^{(j)} \in X_j$ such that $\rho_j\text{-}\lim_{n\to\infty} x_n^{(j)} = y^{(j)}$, $j=1,2$. By (2) of Thm.10 (on p.III.4), $\rho_{sum}\text{-}\lim_{n\to\infty} x_n = y$. Therefore (X, ρ_{sum}) is complete.

The other equivalences are proved similarly. (Alternatively, you may use the fact that a sequence (a_n) is Cauchy w.r.t. ρ_{sum} iff it is Cauchy w.r.t. ρ_{max} iff it is Cauchy w.r.t. ρ_E, which is clear by (1) of Thm.10 on p.III.4.)

<u>p.III.6</u>

<u>Example (1)</u> To see that A is closed, let (a,b) be a limit point of A, and let $(a,b) = \lim_{n\to\infty}(x_n, y_n)$, where $(x_n, y_n) \in A$. Then $3x_n^2 + 5y_n^2 \leq 7 \cdots (1)$ for every $n \in \mathbb{N}$. Because $a = \lim_{n\to\infty} x_n$, $b = \lim_{n\to\infty} y_n$ (by Thm. 10 (2) of p.III.4), we have (by (1)): $3a^2 + 5b^2 \leq 7$. Hence $(a,b) \in A$. Thus A contains all of its limit points, and therefore A is closed.

Alternatively, you may first prove that $\mathbb{R}^2 \setminus A = \{(x,y) \in \mathbb{R}^2 : 3x^2+5y^2 > 7\}$ is open as in Assignment 1 (Prob. VI, no. 2), then deduce that A is closed.

p. III. 6:

Note that $x \in \overline{S}$ iff x is a point of closure of S (as defined on p.I.15)

Pf of this:

Suppose $x \in \overline{S}$. Then $\exists (a_n)_{n \in \mathbb{N}} \subset S$ such that $\lim_{n\to\infty} a_n = x$ (in $\mathbb{R}$). Then $\forall \delta > 0$ $\exists$ $a_n \in S$ such that $|a_n - a| < \delta$; thus $(a-\delta, a+\delta) \cap S \ni a_n$. $\therefore$ $(a-\delta, a+\delta) \cap S \neq \phi$. Therefore a is a point of closure of S according to the definition on p.I. 15. Conversely, suppose a is a point of closure of S according to the definition on p.I.15. Then $\forall k \in \mathbb{N}$, $\phi \neq (a-\frac{1}{k}, a+\frac{1}{k}) \cap S$. Therefore, we may let $a_k \in (a-\frac{1}{k}, a+\frac{1}{k}) \cap S$. It follows that $a_k \in S$ and $|a_k - a| < \frac{1}{k}$; hence $a = \lim_{k\to\infty} a_k$, and a is a limit point of S, i.e. $a \in \overline{S}$.