Pf of (4)

U open, subset of $\mathbb{R}$, $\because = \phi$, nothing to proved. $\phi = (a, a)$

$\phi_n = (n, n) = \phi$, $\bigcup_{n\in\mathbb{N}} \phi_n = \phi$

Suppose $x \in U$, $x \in (a_x, b_x) \subset U$ $\quad B_r(x) = (\underbrace{x-r}_{a_x}, \underbrace{x+r}_{b_x})$

$U = \bigcup_{x\in U} (a_x, b_x)$

z

a_x a_1 b_x b_1

replace $(a_x, b_x) \cup (a_1, b_1)$ by the open interval (a_x, b_1)

$= \bigcup_{i\in I} (a_i, b_i)$ pw disjoint

I countable

U

$\mathbb{R}$

a_i r_i b_i r_j

r_i rational no.

$\exists: I \xrightarrow[1-1]{} \{\text{rational nubers}\}$

$\Rightarrow$ I is countable

Cardinal arithmetric

Thm (⇒) Let S be an open subset of X. To see that $S' \stackrel{def}{=} X \setminus S$ is closed, let t be a lim pt of S', and let $t = \lim_{n\to\infty} t_n$ where $t_n \in S'$. If $t \in X\setminus S' = S$, then (as S is open, $\exists\ \delta > 0$ such that $B_\delta(t) \subset S$

S t S'

As $t = \lim_{n\to\infty} t_n$, $\exists_{N\in\mathbb{N}}$ such that $\rho(t_n, t) < \delta$ whenever $n \geq N$ in particular, $\rho(t_N, t) < \delta$ But this implies that $t_N \in B_\delta(t)$ & therefore $t_N \in S$. This is absurd as $t_N \in S'$. Therefore ~~$t \in S'$~~ $t \notin X\setminus S'$, or $t \in S'$

$\therefore$ S' is closed.

(⇐) Exercise

$$(\underline{A \cup B})' = \underbrace{\underbrace{A'}_{\text{open}} \cap \underbrace{B'}_{\text{open}}}_{\text{open}}$$

Prop. 14 Let $(a_n)_{n\in\mathbb{N}}$ be Cauchy w.r.t. ρ_S in S.

Then (a_n) is also Cauchy in $\underbrace{(X, \rho)}_{\text{complete}}$

$\therefore \exists \ell \in X, \rho(a_n, \ell) \to 0$ as $n \to \infty$

ℓ is a lit pt of S

As S is closed, $\ell \in S$

$\therefore \rho_S(a_n, \underset{\in S}{\ell}) \to 0$ as $n \to \infty$

$\therefore S$ is complete

Ex. 1

Let $(x_n, y_n) \in A$ | $3x_n^2 + 5y_n^2 \le 7$

$\downarrow$ | $\Downarrow$

$(x, y) \in \mathbb{R}^2$ | $3x^2 + 5y^2 \le 7$ i.e. $(x, y) \in A$

$\therefore A$ is closed

$X_1 \times X_2$ ρ is product metric of ρ_1 by ρ_2
ρ_1 ρ_2

closed $A_1 \times A_2$ closed is closed w.r.t ρ in $X_1 \times X_2$

because if $\boxed{(a_{n,1}, a_{n,2}) \in A_1 \times A_2}$ for all $n \in \mathbb{N}$

$\downarrow$

$\boxed{(a_1, a_2) \in X_1 \times X_2}$

By prev. thm, $a_{n,1} \to a_1$, $a_{n,2} \to a_2$ acc. to ρ_1, ρ_2 resp. as $n \to \infty$

Since A_1, A_2 are closed, $a_1 \in A_1$, $a_2 \in A_2$, i.e. $(a_1, a_2) \in A_1 \times A_2$

$S \subset X, \rho$

closure $\overline{S}$ $\quad \overline{S} \overset{\text{def}}{=} \{x \in X : x \text{ is a limit pt. of } S\}$

$X = \mathbb{R}$

0 $\quad$ 1

$S = (0,1)$ is open in $\mathbb{R}$, $\quad \overline{S} = \{x \in \mathbb{R} : x \text{ is a lit pt of } (0,1)\}$

$\overset{?}{=} [0,1]$

$\overline{S} \overset{?}{\supset} [0,1]$

Of course $\boxed{\overline{S} \supset S} = (0,1)$

because for each $s \in S$, consider $s_n = s \ \forall n \in \mathbb{N}$

$s_n \to s$ as $n \to \infty$

$\therefore s \in \overline{S}$

$0, 1 \in \overline{S}$

$\overset{?}{=}$

$\lim\limits_{n\to\infty} \frac{1}{n+1}$

$\overline{S} \subset [0,1]$

$t = \lim\limits_{n\to\infty} s_n$

$\in S = (0,1)$, $0 < s_n < 1$

$\therefore 0 \le \lim\limits_{n\to\infty} s_n \le 1$, i.e. $t \in [0,1]$

$x \in \overline{S}$

$\therefore x = \lim\limits_{n\to\infty} x_n$, i.e. x is a lit pt. of S

$\therefore x_n \in S$, $|x_n - x| < \frac{1}{n}$

$\exists x_n \in$

"$\Leftarrow$" $\forall n \in \mathbb{N}$, $(x - \frac{1}{n}, x + \frac{1}{n}) \cap S \ne \emptyset$

i.e. $B_\varepsilon(x) \cap S \ne \emptyset$

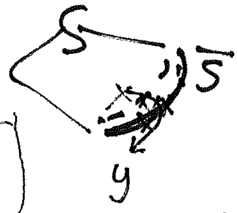
$\Leftarrow \Rightarrow$ $\forall \varepsilon > 0$, $(x-\varepsilon, x+\varepsilon) \cap S \ne \emptyset$

in our approach mean that

$\boxed{x \in \overline{S}}$ iff x is a pt. of closure of S

Prop 15 (1)

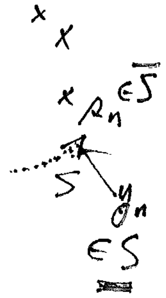
We know $\boxed{\overline{S} \supset S}$ already



$\boxed{\overline{S} \text{ is closed}}$: let y be a lim pt of $\overline{S}$

Want to see that $y \in \overline{S}$ i.e. want $(y_n) \in S$ such that $y = \lim_{n\to\infty} y_n$.

$\exists \, (a_n)_{n\in\mathbb{N}}$ in $\overline{S}$ such that $y = \lim_{n\to\infty} a_n$



As $a_n \in \overline{S}$, $\exists \, y_n \in S$ such that $\rho(a_n, y_n) < \frac{1}{n}$.

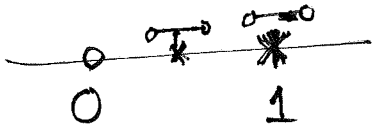
Now $$\rho(y, y_n) \le \underbrace{\rho(y, a_n)}_{\to\, 0 \text{ as } n\to\infty} + \underbrace{\rho(a_n, y_n)}_{<\frac{1}{n} \,\to\, 0 \text{ as } n\to\infty}$$

$\therefore \lim_{n\to\infty} \rho(y, y_n) = 0$ i.e. $y = \lim_{n\to\infty} y_n$

$\overline{S}$ is the smallest such thing

i.e. if T is closed, & $T \supset S$, then $T \supset \overline{S}$

because lim pts of S are also lim pts of T



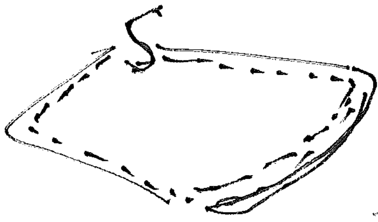
$S \stackrel{\text{def}}{=} (0, 1]$

$(0,1) \subset S^\circ$

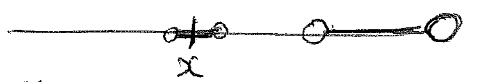
$\forall \varepsilon > 0$, $B_\varepsilon(1) \not\subset (0, 1]$

$\therefore 1 \notin S^\circ$

$(0,1) = S^\circ$

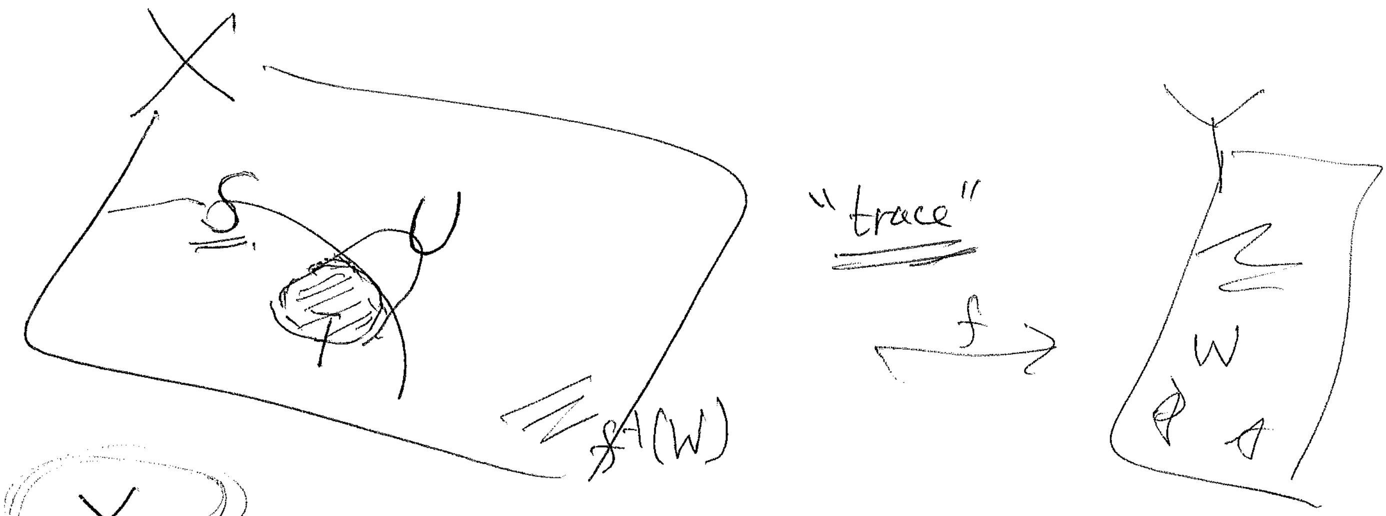


$$\frac{f(x+h) - f(x)}{h}$$



open

lim



X

$\mathcal{T} \subset \mathcal{P}(X)$ — power set of X

each elt of $\mathcal{T}$ is a subset of X

is a set of ~~some~~ some subsets of X

(1) $\phi, X \in \mathcal{T}$

(2) if $U, V \in \mathcal{T}$, then $U \cap V \in \mathcal{T}$

(3) if $(U_i)_{i\in I}$ is a family in $\mathcal{T}$, then $\bigcup_{i\in I} U_i \in \mathcal{T}$

Then $\mathcal{T}$ is called a topology on X
for X.

$(X, \mathcal{T})$.

$\mathcal{T} \quad \mathcal{S}$

$X \xrightarrow{f} Y$

(2) $f^{-1}(W) \in \mathcal{T}$ whenever $W \in \mathcal{S}$

Thm 18

"(1) $\Rightarrow$ (2)"

Suppose f is cont on X, and let W be an open subset of Y.

If ~~$W = \phi$, then~~ $f^{-1}(W) = \phi$, then it is open in X.

Suppose $W \neq \phi$, and let $x \in \boxed{f^{-1}(W)}$. Want $\delta > 0$ such that $\boxed{B_\delta(x) \subset f^{-1}(W)}$.

$x \in f^{-1}(W)$ is open

As $x \in f^{-1}(W)$, so $y \overset{\text{def}}{=} f(x) \in W$ open in Y. $\exists\, \varepsilon > 0$ such that $\boxed{B_\varepsilon(y) \subset W}$. As f is cont at x,

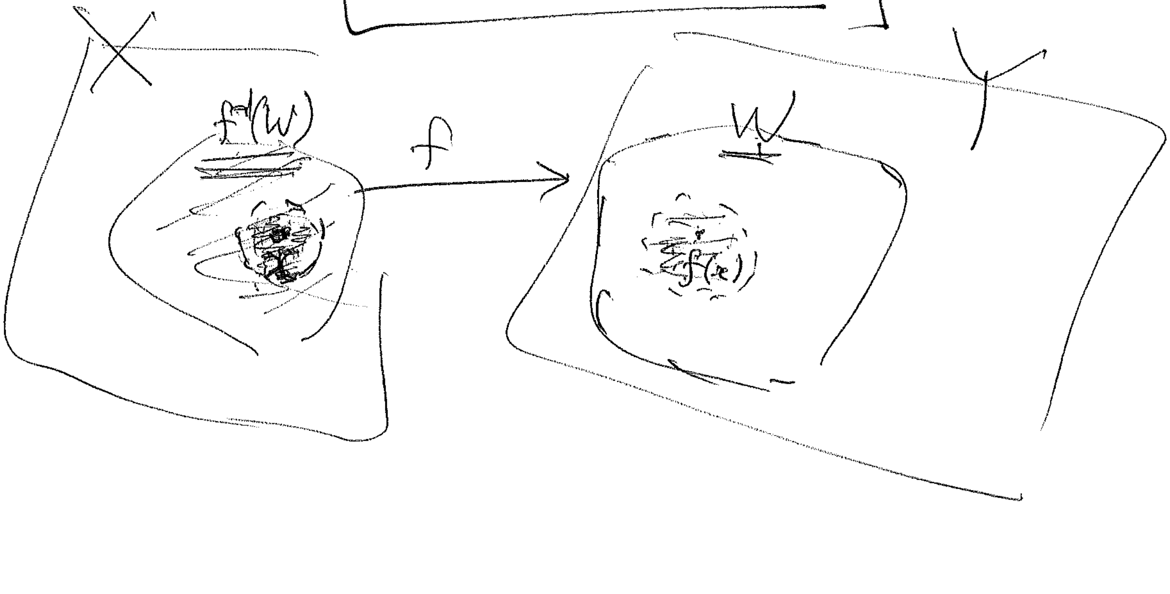
$\exists\, \delta > 0$ such that $\sigma(f(x), f(z)) < \varepsilon$ whenever $\rho(x, z) < \delta$

i.e. $f(z) \in B_\varepsilon(f(x))$ when $z \in B_\delta(x)$

$$B_\varepsilon(f(x)) = B_\varepsilon(y)$$

$\therefore \quad B_\delta(x) \subset f^{-1}(B_\varepsilon(y)) \subset f^{-1}(W)$

$\therefore \quad \boxed{B_\delta(x) \subset f^{-1}(W)}$



(P.III.6) Prop. 15 (1) $\overline{S}$ is the smallest closed set containing S.

Pf Because each $x \in S$ is a limit point of S ($x = \lim_{n\to\infty} x_n$, where $x_n = x$), $S \subset \overline{S}$. To see that $\overline{S}$ is closed, let $y \in X$ be a limit point of $\overline{S}$, i.e. $y = \lim_{n\to\infty} y_n$, where $y_n \in \overline{S}$. By the definition of a limit point, for each $n \in \mathbb{N}$, $\exists z_n \in S$ such that $\rho(y_n, z_n) < \frac{1}{n}$. We'll see that $y = \lim_{n\to\infty} z_n$ (then $y \in \overline{S}$, and $\overline{S}$ is closed). Let $\varepsilon > 0$, $\exists N_1 \in \mathbb{N}$ such that $\rho(y, y_n) < \varepsilon/2$ whenever $n \geq N_1$. Define $N = \max\left(\frac{2}{\varepsilon}, N_1\right)$. Then $\forall n \geq N$,

$$\rho(y, z_n) \leq \rho(y, y_n) + \rho(y_n, z_n) < \frac{\varepsilon}{2} + \frac{1}{n} \leq \frac{\varepsilon}{2} + \frac{1}{N} \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Therefore $y = \lim_{n\to\infty} z_n$ and we are done.

(P.III.7)

Prop. 17 (1)($\Rightarrow$) Suppose T is open in (S, ρ_S). Then $\forall x \in T$, $\exists \varepsilon_x > 0$, $S \cap B_{\varepsilon_x}(x) \subset T$.

Let $U = \bigcup_{x \in T} B_{\varepsilon_x}(x)$, and check that $T = S \cap U$.

Prop. 17 (2) ($\Rightarrow$) Take $A = \overline{T}$ (the closure of T as a subset of X).

Thm. 18 (1) $\Rightarrow$ (2). ~~Suppose~~ suppose $x \in f^{-1}(W)$. Then $f(x) \in W$. Because W is open, $\exists \varepsilon > 0$ such that $B_\varepsilon(f(x)) \subset W$, i.e. if $y \in Y$ and $\sigma(y, f(x)) < \varepsilon$, then $y \in W$. Because f is continuous at x, there exists $\delta > 0$ such that $\sigma(f(x'), f(x)) < \varepsilon$ whenever $x' \in X$ and $\rho(x, x') < \delta$. It follows that $\forall x' \in B_\delta(x)$, $f(x') \in W$, i.e. $x' \in f^{-1}(W)$. Thus $B_\delta(x) \subset f^{-1}(W)$. Therefore $f^{-1}(W)$ is open (w.r.t. ρ).

If $f^{-1}(W)$ is empty, then $f^{-1}(W)$ is open. In case $f^{-1}(W) \neq \phi$,