Since what matters is really the metric, we will go over to metric spaces.

Let X be a (non-empty) set. A map $\rho: X\times X \to \mathbb{R}$ is a <u>metric</u> on X if the following three conditions are satisfied:

(1) $\rho(x,y) \geq 0$ for all $x,y \in X$,
$\rho(x,y) = 0$ iff $x=y$;

(2) $\rho(x,y) = \rho(y,x)$ for all $x,y \in X$;

(3) $\rho(x,z) \leq \rho(x,y) + \rho(y,z)$ for all $x,y,z \in X$.

The pair (X,ρ) is called a <u>metric space</u> if ρ is a metric on X. When no confusion is likely, we suppress ρ, and call X a metric space.

<u>Examples</u>

(1) Our most important examples are $\mathbb{R}^p$, $p \in \mathbb{N}$, with the usual metrics.

(2) Let $X = \{1,2,3\}$. Define ρ on $X\times X$ to $\mathbb{R}$ by:

$$\rho(1,1) = \rho(2,2) = \rho(3,3) = 0;$$

$\rho(x,y) = 1$, if $x,y \in X$ and $x \neq y$.

Then clearly (X,ρ) is a metric space. Moreover, this example can be generalized to an arbitrary (non-empty) set easily.

(3) Let S be a non-empty subset of $\mathbb{R}^p$, and let X be the set of bounded, real-valued functions on S, i.e.

$$X = \left\{ f \,\middle|\, \begin{array}{l} f: S \to \mathbb{R}, \text{ and } \exists\, k_f \in \mathbb{R} \ (k_f \text{ depends on } f) \\ \text{such that } |f(s)| \leq k_f \text{ for all } s \in S \end{array} \right\}.$$

Define $\rho: S \times S \to \mathbb{R}$ by

$$\rho(f, g) = \sup\{|f(s) - g(s)| : s \in S\}$$

(note that $|f(s) - g(s)| \leq |f(s)| + |g(s)| \leq k_f + k_g$ for all $s \in S$).

Then it is easily checked that (X, ρ) is a metric space.

(4) Let (X, ρ) be a metric space, and S a subset of X.

Then $\rho_S : S \times S \to \mathbb{R} : (x, y) \mapsto \rho(x, y)$,

is a metric on S, and (S, ρ_S) is a metric space. By abuse of notation, we write ρ instead of ρ_S. And (S, ρ) is called a sub-metric space, or metric subspace, of (X, ρ).

(5) Let (X_1, ρ_1), (X_2, ρ_2) be metric spaces. Define $X = X_1 \times X_2$ and ρ_E, ρ_{max}, $\rho_{sum} : X \times X \to \mathbb{R}$ by:

$$\rho_E((x_1, x_2), (y_1, y_2)) = \sqrt{\rho_1(x_1, y_1)^2 + \rho_2(x_2, y_2)^2} \quad \left(\begin{array}{c}\text{called the product} \\ \text{metric of } \rho_1 \text{ by } \rho_2\end{array}\right),$$

$$\rho_{max}((x_1, x_2), (y_1, y_2)) = \max\{\rho_1(x_1, y_1), \rho_2(x_2, y_2)\}$$

$$\rho_{sum}\big((x_1,x_2),(y_1,y_2)\big) = \rho_1(x_1,y_1) + \rho_2(x_2,y_2).$$

then (X,ρ_E), (X,ρ_{max}), (X,ρ_{sum}) are metric spaces. Moreover, the usual metric on $\mathbb{R}^2 = \mathbb{R}\times\mathbb{R}$ is ρ_E with $X_1 = X_2 = \mathbb{R}$, $\rho_1 = \rho_2 =$ usual metric on $\mathbb{R}$.

Clearly, the concepts of a convergent sequence, a Cauchy sequence, and completeness can be analogorously generalized (from $\mathbb{R}^p$) to any metric space. And so is the continuity of a map from a metric space to another metric space: Let (X,ρ) be a metric space, $a_n \in X$ for every $n\in\mathbb{N}$, the sequence (a_n) is <u>convergent</u> if $\exists\, l\in X$ such that $\rho\text{-}\lim_{n\to\infty} a_n = l$ (in words, (a_n) converges to l w.r.t. ρ), which means that

$$\forall \varepsilon>0,\ \exists N\in\mathbb{N} \text{ such that } \rho(a_n, l)<\varepsilon \text{ whenever } n\geq N,$$

or equivalently,

$$\forall \varepsilon>0\ \exists N\in\mathbb{R} \text{ such that } \rho(a_n,l)\leq\varepsilon \text{ whenever } n>N;$$

(a_n) is <u>Cauchy</u> if

$$\forall \varepsilon>0\ \exists N\in\mathbb{N} \text{ such that } \rho(a_n,a_p)<\varepsilon \text{ whenever } n,p\geq N;$$

(X,ρ) is <u>complete</u> iff every Cauchy sequence (a_n) in X (i.e. $a_n\in X$ for every $n\in\mathbb{N}$) converges to some $l\in X$; a map $f: X\to Y$ is <u>ρ-σ continuous</u> (or just <u>continuous</u>) at a point $x_0\in X$, where σ is a metric on Y, if $\forall\varepsilon>0\ \exists\delta>0$ such that $\sigma(f(x), f(x_0))<\varepsilon$ whenever $\rho(x,x_0)<\delta$ (and, of course, $x\in X$).

<u>Thm.10</u> In the notation of Ex. (5) above, we have:

(1) for all $x, y \in X$, pf

$$\rho_{max}(x,y) \leq \rho_E(x,y) \leq \rho_{sum}(x,y) \leq 2\,\rho_{max}(x,y).$$

(2) for every sequence $(a_n = (a_{n,1}, a_{n,2}))_{n\in\mathbb{N}}$ in X (i.e. $a_{n,j} \in X_j$, $j=1,2$ for every $n \in \mathbb{N}$) and $l = (l_1, l_2) \in X$ (i.e. $l_j \in X_j$, $j=1,2$),

$$\rho_E\text{-}\lim_{n\to\infty} a_n = l \quad \text{iff} \quad \rho_{max}\text{-}\lim_{n\to\infty} a_n = l$$

$$\text{iff} \quad \rho_{sum}\text{-}\lim_{n\to\infty} a_n = l$$

$$\text{iff} \quad \rho_j\text{-}\lim a_{n,j} = l_j \text{ for } j=1,2.$$

<u>Cor</u> X is complete w.r.t. any one of $\rho_E, \rho_{max}, \rho_{sum}$ iff X_j is complete w.r.t. ρ_j for $j=1,2$, provided $X_j \neq \phi$ for $j=1,2$.

Obviously, Ex. (5) and the above Thm (for two factors) can be generalized to any product of finitely many factors, in particular, for each $p \in \mathbb{N}$, ρ_E on $\mathbb{R}^p$ is the usual metric on $\mathbb{R}^p$.

Let (X, ρ) be a metric space. Then, for $r>0$ and $x \in X$, the <u>ball</u> $B(x;r)$, or $B_r(x)$, <u>centered at x with radius r</u> is defined by

$$B_r(x) = B(x;r) = \{y \in X : \rho(y,x) < r\}.$$

Let $S \subset X$. S is said to be <u>open</u> if for each $x \in S$, there exists $r > 0$ such that $B_r(x) \subset S$. [$y \in S$ whenever $y \in X$, $\rho(y,x) < r$] $c \in X$ is called a <u>limit point</u> of S if there exists a sequence (a_n) in S (i.e. $a_n \in S$ for all $n \in \mathbb{N}$) such that $c = \lim_{n\to\infty} a_n$ (w.r.t. ρ). S is said to be <u>closed</u> if S contains all of its limit points.

<u>Prop. 11</u>

(1) ϕ, X are open.

(2) If U, V are open subsets of X, then $U \cap V$ is open.

(3) If $(U_i)_{i\in I}$ is a family of open subsets of X (i.e., if each U_i, $i \in I$, is an open subset of X), then $\bigcup_{i \in I} U_i$ is open.

<u>Examples</u>

(1) $B_r(x)$ is open.

(2) In $\mathbb{R}$, for all $a, b \in \mathbb{R}$, (a,b), $(-\infty, b)$, (a, ∞) are open.

(3) In $\mathbb{R}^2$, for all $a, b, c, d \in \mathbb{R}$, $(a,b) \times (c,d)$ is open; moreover, for all U, V open subsets of $\mathbb{R}$, $U \times V$ is open.

*(4) Each open subset U of $\mathbb{R}$ can be uniquely expressed as a countable union of disjoint open intervals. The endpoints of these intervals do not belong to U. (cf. pp.63, Thm 9, Ref. 4 [Ref (4)] [Royden: Real Analysis])

Thm. 12 S is open iff its complement $S' \overset{\text{def}}{=} X \setminus S$ in X is closed.

Prop. 13

(1) ϕ, X are closed.

(2) If $A, B \subset X$ are closed, then $A \cup B$ is closed.

(3) If $(A_i)_{i \in I}$ is a family of closed subsets of X, then $\bigcap_{i \in I} A_i$ is closed.

Prop. 14 If (X, ρ) is complete and $S \subset X$ is closed, then (S, ρ_S) is complete.

Examples

(1) In $\mathbb{R}^2$, $A \overset{\text{def}}{=} \{(x,y) \in \mathbb{R}^2 : 3x^2 + 5y^2 \leq 7\}$ is closed.

(2) In the notation of Ex(5) of metric spaces, let (X, ρ) be the product metric space of (X_1, ρ_1) by (X_2, ρ_2). If A_j is a closed subset of X_j, $j = 1, 2$, then $A_1 \times A_2$ is closed in X.

The **closure** $\overline{S}$ of S is defined by: $\overline{S} \overset{\text{def}}{=} \{x : x \text{ is a limit point of } S\}$.
Note that $x \in \overline{S}$ iff x is a point of closure of S (as defined on p. I.15).

Prop. 15

(1) $\overline{S}$ is the smallest (w.r.t. $\subset$) closed set containing S.

(2) $\overline{S} = S$ if and only if S is closed. In particular, $\overline{(\overline{S})} = \overline{S}$.

The <u>interior</u> S° of S is defined by:

$$S^\circ \overset{\text{def}}{=\!=} \{c \in X : \exists \varepsilon > 0,\ B_\varepsilon(c) \subset S\}.$$

An element of S° is called an <u>interior point</u> of S.

<u>Prop. 16</u>

(1) S° is the largest open subset contained in S.

(2) $S^\circ = S$ iff S is open. In particular, $(S^\circ)^\circ = S^\circ$.

<u>Prop. 17</u> Let $T \subset S \subset X$, and let ρ be a metric on X.

(1) T is open in (S, ρ_S) (where ρ_S denotes the metric on S inherited from ρ) iff

$$T = S \cap U$$

for some open subset U of X (w.r.t. ρ).

(2) T is closed in (S, ρ_S) iff

$$T = S \cap A$$

for some closed subset A of X (w.r.t. ρ).

<u>Thm. 18</u> Let (X, ρ) and (Y, σ) be metric spaces, and let $f: X \to Y$ be a map. Then the following statements are equivalent:

(1) f is continuous on X (i.e. f is continuous at every $x \in X$);

(2) for every open subset W of Y, $f^{-1}(W)$ is open (w.r.t. ρ);

(3) for every closed subset B of Y, $f^{-1}(B)$ is closed (w.r.t. ρ).

In the notation of the preceding theorem, if f is bijective and if both f and f^{-1} are continuous (w.r.t. ρ-σ, σ-ρ), then f is called a *homeomorphism* from (X, ρ) to (Y, σ), and (X, ρ) and (Y, σ) are said to be *homeomorphic*.