(2) $\lim\limits_{x\to a} \lambda f(x) = \lambda u$ for each $\lambda \in \mathbb{R}$.

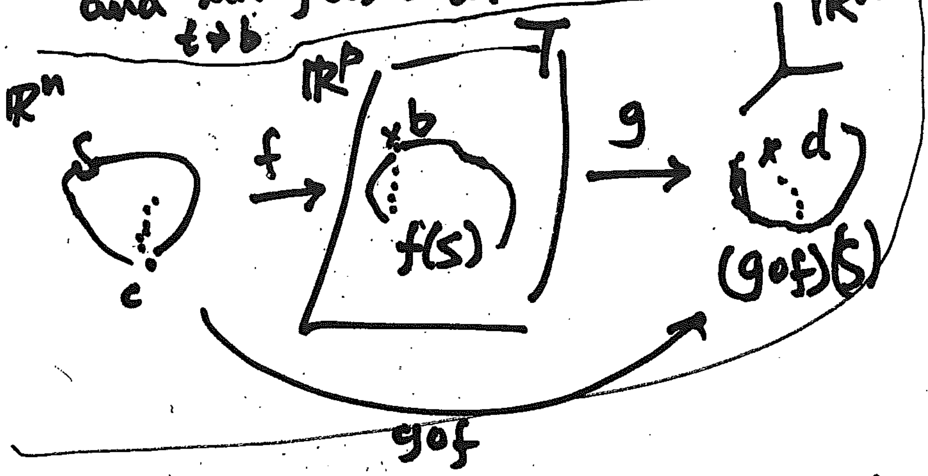
In case $p=1$, we have

(3) $\lim\limits_{x\to a} \left(f(x)g(x)\right) = uv$,

(4) $\lim\limits_{x\to a} \dfrac{f(x)}{g(x)} = \dfrac{u}{v}$ provided $v \neq 0$.

Thm 7 has an obvious corollary in relation to continuity at the point a.

<u>Thm 8</u> Suppose $f: S(\subset \mathbb{R}^n) \to T(\subset \mathbb{R}^p)$, $g: T \to \mathbb{R}^l$. Let c be a pt. of closure of S, b be a pt. of closure of $f(S)$ ($\stackrel{def}{=} \{f(s) : s \in S\}$). Suppose $\lim\limits_{s\to c} f(s) = b$, and $\lim\limits_{t\to b} g(t) = d$.



Then $\lim\limits_{s\to c} (g\circ f)(s) = d$

<u>Pf</u> Let $\varepsilon > 0$. We want to find $\delta > 0$ such that $\|(g\circ f)(s) - d\| < \varepsilon$ whenever $s \in S$ & $\|s - c\| < \delta$.

Because $\lim_{t\to b} g(t) = d$, we have $\delta_1 > 0$ such that

$$(*1) \qquad \| g(t) - d \| < \varepsilon \quad \text{whenever } t \in T \text{ and } \|t - b\| < \delta_1$$

Because $\lim_{s\to c} f(s) = b$, there is $\delta > 0$ such that

$$(*2) \qquad \| f(s) - b \| < \delta_1 \quad \text{whenever } s \in S \text{ and } \|s - c\| < \delta.$$

By $(*1)$ & $(*2)$, we see that for $s \in S$ with $\|s - c\| < \delta$, [using $f(s)$ as t in $(*1)$],

$$\| g(f(s)) - d \| < \varepsilon,$$

i.e. $$\| (g\circ f)(s) - d \| < \varepsilon.$$

$$\therefore \quad \lim_{s\to c} (g\circ f)(s) = d.$$

Cor. Suppose .... Let $c \in S$. Suppose f is cont. at c and g is cont. at $f(c)$. Then $g\circ f$ is cont. at c, i.e.

$$\lim_{s\to c} (g\circ f)(s) = (g\circ f)(c)$$

Ex. Let $v_0 = (x_0, y_0) = (0,0)$, and for $n \in \mathbb{N}$ define

$$v_n = (x_n, y_n) = \left(\sqrt{\frac{x_{n-1}^2 + 2y_{n-1}^2}{4}}, \ \frac{x_{n-1} + y_{n-1} + 1}{3} \right).$$

Show that $\lim_{n\to\infty} v_n$ ~~exists~~ exists, and find its (as a pt. in $\mathbb{R}^2$) its two components.

Thm 9 Let $f: S(\subset \mathbb{R}^q) \to \mathbb{R}^p$, $a \in \mathbb{R}^q$ be a point of

Pf closure of S, and $l \in \mathbb{R}^p$. Then

(1) $\lim_{x \to a} f(x) = l$ iff for every sequence (a_n) in S

with $\lim_{n \to \infty} a_n = a$, $\lim_{n \to \infty} f(a_n) = l$;

$$\Longleftrightarrow$$

(2) $\lim_{x \to a} f(x) = L$ for some $L \in \mathbb{R}^p$ iff the following

condition holds:

(**) for each $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\| f(x) - f(\tilde{x}) \| < \varepsilon$$

whenever $x, \tilde{x} \in S$, $\|x - a\| < \delta$ and $\|\tilde{x} - a\| < \delta$.

$\boxed{11}$ Let us study the topology of $\mathbb{R}^p$. For $c \in \mathbb{R}^p$ and $\varepsilon > 0$,

the <u>open ball</u> $B(c;\varepsilon)$ <u>centered at c with radius $\varepsilon > 0$</u> is

defined by:

$$B(c;\varepsilon) \overset{\text{def}}{=} \{ x \in \mathbb{R}^p : \rho(x,c) < \varepsilon \}.$$

A subset $X \subset \mathbb{R}^p$ is said to be <u>open</u> if $\forall c \in X$, $\exists \varepsilon > 0$,

such that $B(c;\varepsilon) \subset X$.

Pf of Thm 9 (2) ($\Leftarrow$)

Choose $(x_n)_{n\in\mathbb{N}}$ in S such that $\lim_{n\to\infty} x_n = a$. Then by

given $\varepsilon>0$
$\exists \delta>0$
$\vdots$

(**) & ...., $\forall\varepsilon>0$, $\exists M \in \mathbb{N}$ such that

$$\| f(x_n) - f(x_m) \| < \varepsilon \qquad \cancel{\|x_n - a\|<\delta \ \& \ \|x_m - a\|<\delta} \qquad \text{whenever } n, m \geq \cancel{N} M$$

$\therefore$ $(f(x_n))_{n\in\mathbb{N}}$ is Cauchy in $\mathbb{R}^p$

As $\mathbb{R}^p$ is complete, $\exists L \in \mathbb{R}^p$ such that $\lim_{n\to\infty} f(x_n) = L$

It remains to prove that $\lim_{x\to a} f(x) = L$. Let $\varepsilon>0$.

As $\lim_{n\to\infty} f(x_n) = L$, $\exists P \in \mathbb{N}$ such that

$$\| f(x_P) - L \| < \varepsilon/2,$$

and $\quad \| x_P - a \| < \delta \quad$ [δ is given acc. to (**) w.r.t. $\varepsilon/2$]

Then
$$\| f(x) - L \| \leq \| f(x) - f(x_P) \| + \| f(x_P) - L \| < \varepsilon/2 + \varepsilon/2 = \varepsilon,$$

whenever $x \in S$ and $\|x - a\| < \delta$. Thus,

$$\lim_{x\to a} f(x) = L$$