Pf of Thm ($p=2$)

writing $a_n = (x_n, y_n)$

Suppose $(a_n)_{n\in\mathbb{N}}$ (in $\mathbb{R}^2$) satisfies (*). Then, as

$$|x_n - x_m| \leq \sqrt{(x_n-x_m)^2 + (y_n-y_m)^2} = \|a_n - a_m\|,$$

we have

$$(*') \qquad |x_n - x_m| < \varepsilon \text{ whenever } n, m \geq N.$$

i.e. $(x_n)_{n\in\mathbb{N}}$ is Cauchy in $\mathbb{R}^1$. By completeness of $\mathbb{R}^1$, $\exists\, l_1 \in \mathbb{R}$ such that $\lim_{n\to\infty} x_n = l_1$. ($\uparrow 1$)

Similarly, $(y_n)_{n\in\mathbb{N}}$ is a Cauchy seq. in $\mathbb{R}^1$, so $\exists$ $l_2 \in \mathbb{R}$ such that $\lim_{n\to\infty} y_n = l_2$. ($\uparrow 2$)

We'll see that $\lim_{n\to\infty}(x_n, y_n) = (l_1, l_2)$. Let $\varepsilon > 0$ be given. By ($\uparrow 1$), ($\uparrow 2$), $\exists\, N_1, N_2 \in \mathbb{N}$ such that

$$|x_n - l_1| < \frac{\varepsilon}{2} \quad \text{whenever } n \geq N_1,$$

and

$$|y_n - l_2| < \frac{\varepsilon}{2} \quad \text{''} \quad n \geq N_2.$$

Let $P \overset{\text{def}}{=} \max(N_1, N_2)$. Then, for $n \geq P$,

$$\|(x_n, y_n) - (l_1, l_2)\| = \sqrt{(x_n - l_1)^2 + (y_n - l_2)^2}$$

$$\leq \sqrt{\frac{\varepsilon^2}{4} + \frac{\varepsilon^2}{4}} = \frac{\varepsilon}{\sqrt{2}} < \varepsilon$$

$\therefore \lim_{n\to\infty}(x_n, y_n) = (l_1, l_2)$. We conclude that $(a_n)_{n\in\mathbb{N}}$ conv. to (l_1, l_2) in $\mathbb{R}^2$.

$\boxed{10}$ Consider a function $f : S \to \mathbb{R}^p$, where $\phi \neq S \subset \mathbb{R}^n$, $n, p \in \mathbb{N}$. A point $a \in \mathbb{R}^n$ is called a <u>point of closure of S</u> if for all $\delta > 0$, $\{x \in \mathbb{R}^n : \|x - a\| < \delta\} \cap S \neq \phi$. We write $\lim_{x \to a} f(x) = l$, where $l \in \mathbb{R}^p$, and say: <u>$f(x)$ converges to l as x converges to a</u> if for each $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\|f(x) - l\| \overset{\leq}{<} \varepsilon \quad \text{whenever } x \in S \text{ and } \|x - a\| < \delta.$$

(Note that we use the same norm notation $\| \ \|$ for the norms on $\mathbb{R}^n$ and on $\mathbb{R}^p$.) f is said to be <u>continuous at a point $x_0 \in S$</u> if $\lim_{x \to x_0} f(x) = f(x_0)$; equivalently, if for each $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\|f(x) - f(x_0)\| < \varepsilon \quad \text{whenever } x \in S \text{ and } \|x - x_0\| < \delta.$$

f is said to be <u>continuous on S</u> if f is continuous at every point of S. f is said to be <u>discontinuous</u> at $x_0 \in S$ [resp., on S] if f is not continuous at x_0 [resp., on S].

Since $f(x) \in \mathbb{R}^p$, it has p components, so there are functions $f_j : S \to \mathbb{R}$, $j = 1, 2, \cdots, p$, such that $f(x) = (f_1(x), \cdots, f_p(x))$; likewise $l = (l_1, \cdots, l_p)$ with $l_j \in \mathbb{R}$ for $j = 1, 2, \cdots, p$. We write $f = (f_1, \cdots, f_p)$.

Prop. 5 $f=(f_1,\cdots,f_p): S \to \mathbb{R}^p$ is continuous at $x_0 \in S$ iff every $f_j: S \to \mathbb{R}$, $j=1,2,\cdots,p$, is continuous at x_0. Moreover, $\lim\limits_{x\to a} f(x) = \underset{(l_1,\cdots,l_p)}{\overset{}{\underset{\|}{l}}}$ iff $\lim\limits_{x\to a} f_j(x) = l_j$ for every $j=1,2,\cdots,p$.

For a function $f: S \longrightarrow \mathbb{R}$ (i.e. for $p=1$), $a \in S \subset \mathbb{R}^n$, we may define $\lim\limits_{x\to a} f(x) = \pm\infty$ as before (when $n=1$).

Example 1 Each elementary function, e.g. a polynomial of several real variables, trigonometric polynomial, exponential function, logarithm function, etc., is continuous on its domain.

Example 2 Let $l \in \mathbb{R}$ and let $f: \mathbb{R} \to \mathbb{R}$ be given by

$$f(x) = \begin{cases} \sin\frac{1}{x}, & \text{if } x \neq 0 \\ l, & \text{if } x = 0. \end{cases}$$

Then f is discontinuous at 0, but continuous at every $x \in \mathbb{R}\setminus\{0\}$. (Note that for $t = \sin\theta$, $f(x_n) = t$ if $x_n = \frac{1}{\theta + 2n\pi}$.)

Example 3 Let $f: \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$f(x,y) = \begin{cases} \dfrac{x^2}{x^2+y^2}, & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

Then $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ does not exist, because for every real number k, $f(x, kx) = \dfrac{x^2}{x^2+k^2x^2} = \dfrac{1}{1+k^2}$ for $x \in \mathbb{R}\setminus\{0\}$.

($\forall \delta > 0$, $\|(\frac{2\delta}{3}, \frac{\delta}{2})\| = \frac{5}{6}\delta < \delta$, but

$$\left| f(\tfrac{2\delta}{3}, \tfrac{\delta}{2}) - f(0,0) \right| = \left(\frac{6}{5\delta}\cdot\frac{2\delta}{3}\right)^2 = \frac{16}{25} > \frac{1}{2}.$$

Similar argument shows that no matter what $f(0,0)$ is/undefined, $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ does not exist. The situation is also clear if we employ polar coordinates.)

Thm 6 The following are equivalent:

(1) f is continuous on S,

(2) for every convergent sequence $(x_n) \to a \in S$ and $x_n \in S$,

$$\lim_{n\to\infty} f(x_n) = f(a),$$

where $\phi \neq S \subset \mathbb{R}^n$, $f: S \to \mathbb{R}^p$.

Thm 7 Let $f, g: S \to \mathbb{R}^p$, $S \subset \mathbb{R}^n$, a is a point of closure of S. Suppose $\lim\limits_{x\to a} f(x) = u$, $\lim\limits_{x\to a} g(x) = v$, where $u, v \in \mathbb{R}^p$. Then

(1) $\lim\limits_{x\to a} (f(x) + g(x)) = u + v$,

(2) $\lim_{x \to a} \lambda f(x) = \lambda u$ for each $\lambda \in \mathbb{R}$.

In case $p=1$, we have

(3) $\lim_{x \to a} \left(f(x) g(x) \right) = uv$,

(4) $\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{u}{v}$ provided $v \neq 0$.

Thm 7 has an obvious corollary in relation to continuity at the point a.