$\boxed{5}$ The order completeness of $\mathbb{R}$ has a very important consequence: the metric completeness (in fact, these two kinds of completeness are equivalent). To explain this, observe that if a sequence (a_n) converges (in $\mathbb{R}$), then

$$(*)\quad \left[\begin{array}{l} \text{for each } \varepsilon > 0, \text{ there exists } N \in \mathbb{N} \text{ such that} \\ \qquad\qquad |a_p - a_n| < \varepsilon \\ \text{whenever } p, n \geq N. \end{array}\right.$$

(a_n) is called a <u>Cauchy sequence</u> if (a_n) has the above property $(*)$. So, every convergent sequence is a Cauchy sequence. $\mathbb{R}$ is <u>metrically complete</u> in the sense that the following theorem holds.

<u>**Thm. 4**</u> (metric completeness)

Every Cauchy (real) sequence converges in $\mathbb{R}$.

For a proof of this theorem (basing on the order completeness), we observe the following:

(1) A Cauchy sequence is bounded.

(2) If a Cauchy sequence $(a_n)_{n\in\mathbb{N}}$ has a convergent subsequence $(a_{n_k})_{k\in\mathbb{N}}$, then (a_n) is convergent.

($(a_{n_k})_{k\in\mathbb{N}}$ is a subsequence of $(a_n)_{n\in\mathbb{N}}$ if $n_k\in\mathbb{N}$, and $n_k < n_{k+1}$, for every $k\in\mathbb{N}$.)

(3) If $[a_1, b_1] \supset [a_2, b_2] \supset \cdots \supset [a_n, b_n] \supset [a_{n+1}, b_{n+1}] \supset \cdots$ and if $\lim_{n\to\infty} (b_n - a_n) = 0$, then there exists exactly one point ξ which belongs to all $[a_n, b_n]$, $n\in\mathbb{N}$. (This involves the order completeness.)

(4) Any bounded (real) sequence has a convergent subsequence. [This is hard to prove.]

Below we give proofs of (2) and (4).

Ad (2) Let $(a_{n_k})_{k\in\mathbb{N}}$ be a convergent subsequence of a Cauchy sequence $(a_n)_{n\in\mathbb{N}}$. Let $\varepsilon>0$, and let ℓ be the limit of $(a_{n_k})_{k\in\mathbb{N}}$. Then

(**) $\exists\, N\in\mathbb{N}$ such that $|a_n-a_p|<\frac{\varepsilon}{2}$ whenever $n,p\geq N$,

(†) $\exists\, K\in\mathbb{N}$ such that $|a_{n_k}-\ell|<\frac{\varepsilon}{2}$ whenever $k\geq K$.

Let $k_1\in\mathbb{N}$ be such that $k_1\geq K$ and $n_{k_1}\geq N$.

Then, for $n\geq N$,

$$|a_n-\ell|\leq |a_n-a_{n_{k_1}}|+|a_{n_{k_1}}-\ell|$$

$$<\frac{\varepsilon}{2}+\frac{\varepsilon}{2}\quad(\text{by } (**)\ \&\ (\dagger))=\varepsilon.$$

$\therefore\ |a_n-\ell|<\varepsilon$ whenever $n\geq N$.

$\therefore\ (a_n)_{n\in\mathbb{N}}$ converges to ℓ.

Ad (4) Let $(c_n)_{n\in\mathbb{N}}$ be a bounded real sequence. As explained in class, there are intervals $[a_k,b_k]$, $k\in\mathbb{N}$, positive integers $n_1<n_2<\cdots<n_k<n_{k+1}<\cdots$, such that (*)

$$[a_1, b_1] \supset [a_2, b_2] \supset \cdots \supset [a_k, b_k] \supset [a_{k+1}, b_{k+1}] \supset \cdots,$$
$$\quad c_{n_1} \in [a_1,b_1],\quad c_{n_2} \in [a_2,b_2],\quad c_{n_k} \in [a_k,b_k],\quad c_{n_{k+1}} \in [a_{k+1},b_{k+1}]$$

and $b_k - a_k = \frac{1}{2^{k-1}}(b_1 - a_1)$, for all $k \in \mathbb{N}$.

According to (3), let $\xi \in [a_k, b_k]$ for all $k \in \mathbb{N}$. Then clearly

$$|c_{n_k} - \xi| \leq b_k - a_k \qquad \text{for all } k \in \mathbb{N}.$$

It follows that $\lim_{k \to \infty} c_{n_k} = \xi$.

<u>Alternative proof of Thm 4</u>

Since (a_n) is Cauchy, $\exists N_1 \in \mathbb{N}$ such that

$$|a_n - a_{N_1}| < 1 \quad \text{whenever } n \geq N_1,$$

i.e. $a_{N_1} - 1 < a_n < a_{N_1} + 1$ for all $n \geq N_1$. Let $M \in (0, \infty)$ be such that $a_1, a_2, \cdots, a_{N_1 - 1}, a_{N_1} - 1, a_{N_1} + 1 \in [-M, M]$.

Then $\{a_n : n \in \mathbb{N}\} \subset [-M, M]$.

(*) Let $a_1 = -M$, $b_1 = M$, $n_1 = 1$. Let $I_0 = [a_1, \frac{1}{2}(a_1+b_1)]$, $I_1 = [\frac{1}{2}(a_1+b_1), b_1]$. If $\exists$ infinitely many n such that $c_n \in I_0$, let $[a_2, b_2] = I_0$; otherwise, let $[a_2, b_2] = I_1$. Let $n_2 \in \mathbb{N}$ be such that $n_2 > n_1$ and $c_{n_2} \in [a_2, b_2]$ (this is possible, because $\exists$ infinitely many n such that $c_n \in [a_2, b_2]$).

Define $S = \{s \in [-M, M] : \{n \in \mathbb{N} : a_n \geqslant s\} \text{ is infinite}\}$. Then $-M \in S$ and S is bounded above by M. By the order completeness of $\mathbb{R}$, let $b = \sup S$.

We claim that $\lim_{n\to\infty} a_n = b$. To this end, let $\varepsilon > 0$. Then, as (a_n) is a Cauchy sequence,

(i) $\exists N_2 \in \mathbb{N}$ such that $|a_p - a_n| < \frac{\varepsilon}{2}$ whenever $p, n \geqslant N_2$;

as $b = \sup S$, $b + \frac{\varepsilon}{2} \notin S$, $\{n \in \mathbb{N} : a_n \geqslant b + \frac{\varepsilon}{2}\}$ is finite,

(ii) $\exists N_3 \in \mathbb{N}$ such that $N_3 \geqslant N_2$ and

$a_n < b + \frac{\varepsilon}{2}$ whenever $n \geqslant N_3$;

as $b = \sup S$, $\exists s_0 \in S$ (i.e. $\{n \in \mathbb{N} : a_n \geqslant s_0\}$ is infinite) satisfying $s_0 > b - \frac{\varepsilon}{2}$,

(iii) $\exists N \geqslant N_3$ such that $b - \frac{\varepsilon}{2} < (s_0 \leqslant)\, a_N$. By (ii) we also have $a_N < b + \frac{\varepsilon}{2}$. Thus $|a_N - b| < \frac{\varepsilon}{2}$.

As $N \geqslant (N_3 \geqslant) N_2$, by (i) and (iii),

$$|a_n - b| \leqslant |a_n - a_N| + |a_N - b| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

whenever $n \geqslant N$. Therefore

$$\lim_{n\to\infty} a_n = b.$$

$\boxed{6}$ Let $f: A \to B$, where A, B are non-empty subsets of $\mathbb{R}$. Let $x_0 \in \mathbb{R}$ be such that for all $\delta > 0$,

$$(x_0 - \delta, x_0 + \delta) \cap A \neq \phi.$$

Such an x_0 is called a point of closure of A. (This requirement on x_0 is to make the following definition meaningful.)

Def Let f, A, B, x_0 be as above, and let $l \in \mathbb{R}$. Then we say that $f(x)$ converges to l as x converges to x_0, in symbol "$f(x) \to l$ as $x \to x_0$", or "$\lim_{x \to x_0} f(x) = l$", if for each $\varepsilon > 0$, there exists $\delta > 0$ such that

$$|f(x) - l| < \varepsilon \quad \text{whenever} \quad x \in (x_0 - \delta, x_0 + \delta) \cap A.$$

(Note carefully the difference of this definition with what you may have encountered before.)

We write $\lim_{x \to x_0} f(x) = \infty$ [resp., $\lim_{x \to x_0} f(x) = -\infty$] if for each $r \in \mathbb{R}$, there exists $\delta > 0$ such that

$$f(x) > r \quad [\text{resp. } f(x) < r] \text{ whenever } x \in (x_0 - \delta, x_0 + \delta) \cap A$$

Suppose A is not bounded above (in $\mathbb{R}$, i.e. as a subset of $\mathbb{R}$). Then we write $\lim_{x\to\infty} f(x) = \begin{matrix} l \in \mathbb{R} \\ \pm\infty \end{matrix}$ if the corresponding condition with $(x_0-\delta, x_0+\delta)$ replaced by (δ, ∞), is satisfied.

Similarly we define $\lim_{x\to-\infty} f(x) = \begin{matrix} l \in \mathbb{R} \\ \pm\infty \end{matrix}$ in case A is not bounded below.

We have thm & prop similar to those in $\boxed{2}$ and $\boxed{3}$ above.

For examples of limits, please read Reading Material 2 up-loaded in our homepage.

$\boxed{\text{I}}$ We consider some examples.

Ex. 1 Let $\varepsilon>0$, and let $N=\max\{3, \frac{27}{2\varepsilon}\}$. Then for every $n \geq N$,

$$\left|\frac{3^n}{n!}-0\right|=\frac{3\cdot 3\cdot 3\cdots\cdot 3\cdot 3}{1\cdot 2\cdot 3\cdots(n-1)\,n} \leq \frac{3\cdot 3\cdot 3}{1\cdot 2\cdot n}=\frac{27}{2n}\leq \varepsilon.$$

$$\leq \frac{27}{2N} \leq \varepsilon$$

Hence, $$\lim_{n\to\infty}\frac{3^n}{n!}=0.$$

Ex. 2 Let $x\in[0,1]$, and let a sequence $(y_n)_{n\in\mathbb{N}}$ be defined by:

$$y_1=x,\quad y_2=x-\tfrac{1}{2}y_1^2,\quad y_3=x-\tfrac{1}{2}y_2^2,\ \cdots,\ y_{n+1}=x-\tfrac{1}{2}y_n^2.$$

As shown in Ex. 8, p. 4, Reading material 2, we have (as $y_n\in[0,x]$, $y_{2n-1}\geq y_{2n+1}$, $y_{2n}\leq y_{2n+2}$, $y_{2n}\leq y_{2n-1}$):

$$(\star)\quad 0\leq y_2\leq y_4\leq\cdots\leq y_{2n}\leq y_{2n+2}\leq$$
$$\cdots\leq y_{2n+1}\leq y_{2n-1}\leq\cdots y_5\leq y_3\leq y_1=x$$

$$(\circledast)\quad \lim_{n\to\infty}y_{2n-1}=\lim_{n\to\infty}y_{2n}=l=-1+\sqrt{2x+1}=\lim_{p\to\infty}y_p.$$

We now show that $(\circledast)$ holds by an alternative method using the ε-N definition of a limit. To this end, let

$\alpha = \inf\{y_{2n-1} : n \in \mathbb{N}\}$, $\beta = \sup\{y_{2n} : n \in \mathbb{N}\}$.

Then by (*):

① $y_{2n-1} \geq \alpha \geq \beta \geq y_{2n}$ for all $n \in \mathbb{N}$.

Because
$$y_p - y_{p+1} = \frac{y_p^2 - y_{p-1}^2}{2},$$

we have
$$y_{2n-1} - y_{2n} = \frac{1}{2}(y_{2n-1}^2 - y_{2n-2}^2) = \frac{1}{2}(y_{2n-1} + y_{2n-2})(y_{2n-1} - y_{2n-2})$$
$$\leq y_3(y_{2n-1} - y_{2n-2}) = y_3 \cdot \frac{1}{2} \cdot (-y_{2n-2}^2 + y_{2n-3}^2)$$
$$= y_3 \cdot \frac{1}{2}(y_{2n-3} + y_{2n-2})(y_{2n-3} - y_{2n-2})$$

$\therefore$ $y_{2n-1} - y_{2n} \leq y_3^2(y_{2n-3} - y_{2n-2})$;

iterating, we obtain:
$$y_{2n-1} - y_{2n} \leq y_3^{2n-4}(y_3 - y_4)$$

② $\therefore$ $y_{2n-1} - y_{2n} \leq y_3^{2n-3}$

Now, by direct substitution,
$$y_3 = x - \frac{1}{2}x^2 + \frac{1}{2}x^3 - \frac{1}{8}x^4,$$

hence
$$y_3 - \left(1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{8}\right) = (x-1)\left[1 - \frac{1}{2}(x+1) + \frac{1}{2}(1+x+x^2) - \frac{1}{8}(1+x+x^2+x^3)\right]$$
$$= (x-1)\frac{1}{8}\left[(7+3x^2) - x(1+x^2)\right] \leq 0 \quad (\text{as } x \in [0,1]),$$

(II.2)

and thus $y_3 \leq 7/8$. Therefore $y_3^{2n-3} \leq \left(\frac{7}{8}\right)^{2n-3} < \varepsilon$ if $2n-3 > \ln\left(\frac{8}{7\varepsilon}\right)$.

③

By ①, ②, and ③,

(★★) each of $|y_{2n-1}-\alpha|$, $|\alpha-\beta|$, and $|\beta-y_{2n}|$
$< \varepsilon$ if $2n-3 > \ln(\frac{8}{7\varepsilon})$.

Define $N = \ln(\frac{8}{7\varepsilon})+3$. Then $N \in \mathbb{R}$, and it follows from (★★) that

$$|\alpha-\beta|=0, \text{ i.e. } \alpha=\beta,$$

$$\lim_{n\to\infty} y_{2n-1} = l = \lim_{n\to\infty} y_{2n} \quad (\text{where } l \overset{\text{def}}{=} \alpha=\beta),$$

and $|y_p - l| < \varepsilon$ if $p > N$.

Therefore $\lim_{p\to\infty} y_p = l$.

Because $y_{n+1} = x - \frac{1}{2}y_n^2$, we have (letting $n\to\infty$)

$$l = x - \frac{1}{2}l^2,$$

$$l = -1+\sqrt{2x+1} \quad (\text{as } l \geq 0 \text{ by } (\star)).$$

<u>Ex. 3</u> Let $\varepsilon>0$, we show that there exists $\delta>0$ such that $|x^3-2|<\varepsilon$ whenever $|x-\sqrt{2}|<\delta$ (and $x\in\mathbb{R}$).

Indeed, let $\delta = \min(\frac{\varepsilon}{3\sqrt{2}}, \sqrt{2})$. Suppose $|x-\sqrt{2}|<\delta$.

Then $\sqrt{2}-\delta < x < \sqrt{2}+\delta$, $2\sqrt{2}-\delta < x+\sqrt{2} < 2\sqrt{2}+\delta$;

as $\delta \leq \sqrt{2}$, we have $0 < x+\sqrt{2} < 3\sqrt{2}$. Therefore,

$$|x^2-2| = |(x-\sqrt{2})(x+\sqrt{2})| = |x+\sqrt{2}|\,|x-\sqrt{2}|$$
$$\leq 3\sqrt{2}\,|x-\sqrt{2}| < 3\sqrt{2}\,\delta \leq \varepsilon .$$

Thus,
$$|x^2-2| < \varepsilon \quad \text{whenever} \quad |x-\sqrt{2}| < \delta.$$

(This proves that $\lim\limits_{x\to\sqrt{2}} x^2 = 2$.)

<u>Ex. 4</u> Let $\varepsilon > 0$, we show that there exists $\delta > 0$ such that
$$\left|\frac{\sin x}{x} - 1\right| < \varepsilon \quad \text{whenever} \quad 0 < |x| < \delta.$$
(This proves that $\lim\limits_{x\to 0} \frac{\sin x}{x} = 1$.)

First we show that for $x \in [0, \infty)$,
$$(\star) \qquad x - x^2 \leq \sin x \leq x + x^2.$$
Indeed, letting $f(x) = \sin(x) - x + x^2$, $x \in [0,\infty)$, we have $f(0) = 0$, $f'(x) = \cos(x) - 1 + 2x$, $f'(0) = 0$,
$f''(x) = -\sin(x) + 2 > 0$. Hence $f'(x)$ is strictly increasing on $[0,\infty)$, and $f'(x) \geq 0$ for all $x \in [0,\infty)$; it follows that $f(x) \geq 0$ on $[0,\infty)$. Therefore,
$$x - x^2 \leq \sin x.$$
Similarly we have $\sin x \leq x + x^2$ also.

Secondly, from ⊛ we obtain for $x \in (0, \infty)$,

$$-x \leq \frac{\sin x}{x} - 1 \leq x,$$

i.e. $$\left|\frac{\sin x}{x} - 1\right| \leq x.$$

Since $\dfrac{\sin(-x)}{-x} = \dfrac{\sin x}{x}$ for $x \in \mathbb{R} \setminus \{0\}$, we have

$$\left|\frac{\sin x}{x} - 1\right| \leq |x| \quad \text{for all } x \in \mathbb{R} \setminus \{0\}.$$

Let $\delta = \varepsilon$. Then for all $x \in \mathbb{R} \setminus \{0\}$ satisfying $|x - 0| < \delta$,

$$\left|\frac{\sin x}{x} - 1\right| \leq |x| < \delta = \varepsilon,$$

i.e. $\left|\dfrac{\sin x}{x} - 1\right| < \varepsilon$ whenever $0 < |x| < \delta$.

<u>Ex. 5</u> We'll show that $\lim_{n\to\infty} \frac{\ln n}{n} = 0$ according to the ε-N definition

(By calculating $\frac{\ln n}{n}$ for many $n \in \mathbb{N}$, we come to the guess: $\lim_{n\to\infty} \frac{\ln n}{n} = 0$.)

We have for $n \in \mathbb{N}$,

$$\ln n = \int_1^n \frac{1}{x}\,dx < \int_1^n \frac{1}{\sqrt{x}}\,dx = 2(\sqrt{n}-1)$$

$$< 2\sqrt{n};$$

thus $\left|\frac{\ln n}{n} - 0\right| = \frac{\ln n}{n} < \frac{2\sqrt{n}}{n}$.

So, for any given $\varepsilon > 0$, we need only

$$\frac{2\sqrt{n}}{n} < \varepsilon$$

which is equivalent to $n > \frac{4}{\varepsilon^2}$.

Thus, we let $N \stackrel{\text{def}}{=} \frac{4}{\varepsilon^2}$. Then for all $n > N$,

$$\left|\frac{\ln n}{n} - 0\right| < \varepsilon.$$

This shows that $\lim_{n\to\infty} \frac{\ln n}{n} = 0$.

If $\varepsilon = 10^{-2}$, then our $N = 4 \cdot 10^4 = 40000$.

<u>Ex. 6</u> Let a_1, a_2 be positive (real) numbers, and let $a_{n+2} \stackrel{\text{def}}{=} \sqrt{a_{n+1}} + \sqrt{a_n}$ for all $n \in \mathbb{N}$. Find $\lim_{n\to\infty} a_n$ according to the ε-N definition.

<u>Soln</u> We proceed by steps.

(a) $\exists\, M \in \mathbb{N}$ such that $a_n \geq 1$ for all $n \geq M$.

First we observe that $a_n \geq 1$ for all $n \geq P$, if $a_p \geq 1$ and $P \geq 2$ (because $a_{p+1} = \sqrt{a_p} + \sqrt{a_{p-1}} \geq \sqrt{a_p}$). Now suppose (a) is false.

Then $a_n < 1$ for all $n \geq 2$. It follows that for $n \geq 3$,

$$a_n = \sqrt{a_{n-1}} + \sqrt{a_{n-2}} > \sqrt{a_{n-1}} > a_{n-1} \quad (\text{because } a_{n-1} < 1),$$

for $n \geq 2$: $a_{2n} = \sqrt{a_{2n-1}} + \sqrt{a_{2n-2}} > 2\sqrt{a_{2n-2}} > 2\, a_{2n-2}$

hence $a_{2n} > 2\, a_{2n-2} > 2^2 a_{2n-4} > \cdots > 2^{n-1} a_2$, i.e. $a_{2n} > 2^{n-1} a_2$

for all $n \geq 2$. This is absurd because $a_2 > 0$ and $a_{2n} < 1$ for all $n \in \mathbb{N}$

Therefore (a) is established.

(b) Let $\varepsilon_n \stackrel{\text{def}}{=} |a_n - 4|$. Then $\varepsilon_{n+2} \leq \frac{1}{3}(\varepsilon_{n+1} + \varepsilon_n)$ for all $n \geq M$

Pf. We have

$$a_{n+2} - 4 = \sqrt{a_{n+1}} - 2 + \sqrt{a_n} - 2 = \frac{a_{n+1} - 4}{\sqrt{a_{n+1}} + 2} + \frac{a_n - 4}{\sqrt{a_n} + 2},$$

hence

$$|a_{n+2} - 4| \leq \frac{|a_{n+1} - 4|}{\sqrt{a_{n+1}} + 2} + \frac{|a_n - 4|}{\sqrt{a_n} + 2}$$
$$\leq \frac{|a_{n+1} - 4|}{3} + \frac{|a_n - 4|}{3} \quad (\text{by (a)})$$

for all $n \geq M$. Therefore (b) is proved.

(c) Let $d \stackrel{\text{def}}{=} \max\{\varepsilon_M, \varepsilon_{M+1}\}$. Then $\varepsilon_{M+2p+\ell} \leq \left(\frac{2}{3}\right)^p d$ for all $p \geq 1, \ell \geq 0$.

(which implies that $\lim_{n\to\infty} \varepsilon_n = 0$.)

By (b) we get:

$$\varepsilon_{M+2} \leq \tfrac{2}{3} d$$
$$\varepsilon_{M+3} \leq \tfrac{1}{3}\left(\tfrac{2}{3}d + d\right) < \tfrac{1}{3}(d+d) = \tfrac{2}{3} d$$

Similarly,

$$\varepsilon_{M+4} < \tfrac{1}{3}\left(\tfrac{2}{3}d + \tfrac{2}{3}d\right) = \left(\tfrac{2}{3}\right)^2 d$$
$$\varepsilon_{M+5} < \tfrac{1}{3}\left[\left(\tfrac{2}{3}\right)^2 d^2 + \tfrac{2}{3} d\right] < \tfrac{1}{3}\left[\left(\tfrac{2}{3}\right)d + \tfrac{2}{3}d\right] = \left(\tfrac{2}{3}\right)^2 d$$
$$\vdots$$
$$\varepsilon_{M+2p} \leq \left(\tfrac{2}{3}\right)^p d, \quad \varepsilon_{M+2p+1} \leq \left(\tfrac{2}{3}\right)^p d \quad \text{for all } p \in \mathbb{N}$$

It follows that $\varepsilon_{M+2p+\ell} \leq \left(\frac{2}{3}\right)^p d$ for $p \geq 1$, $\ell \geq 0$.

$\boxed{7}$ For each $n \in \mathbb{N}$,

$$\mathbb{R}^n \stackrel{\text{def}}{=\!=} \{(x_1, x_2, \cdots, x_n) : x_j \in \mathbb{R} \text{ for every } j = 1, 2, \cdots, n\}$$

is called the <u>Euclidean n-space</u>. Two n-tuples $(x_1, x_2, \cdots, x_n)$ and $(y_1, y_2, \cdots, y_n)$ are equal iff $x_j = y_j$ for every $j = 1, 2, \cdots, n$. We identify $\mathbb{R}^1$ with $\mathbb{R}$ in an obvious manner: $(x_1) \leftrightarrow x_1$.

There are two operations $+, \cdot$ on $\mathbb{R}^n$, e.g.

$$+: \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n : (x, y) \mapsto x+y,$$

$$\cdot : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n : (\lambda, x) \mapsto \lambda \cdot x,$$

where $x+y \stackrel{\text{def}}{=\!=} (x_1+y_1, x_2+y_2, \cdots, x_n+y_n)$,

$\lambda \cdot x \stackrel{\text{def}}{=\!=} (\lambda x_1, \lambda x_2, \cdots, \lambda x_n)$,

for $x = (x_1, x_2, \cdots, x_n)$, $y = (y_1, y_2, \cdots, y_n) \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}$.

Together with $+$ and $\cdot$, $\mathbb{R}^n$ is a <u>linear space over $\mathbb{R}$</u> (or, a real linear space), i.e. it has the following properties:

(1) $x + y = y + x$ for all $x, y \in \mathbb{R}^n$;

(2) $(x+y)+z = x+(y+z)$ for all $x, y, z \in \mathbb{R}^n$;

(3) with $\mathbf{0} \stackrel{\text{def}}{=\!=} (0, 0, \cdots, 0) \in \mathbb{R}^n$ (called the zero element in $\mathbb{R}^n$),

the zero real number

$x + \mathbf{0} = x$ for all $x \in \mathbb{R}^n$;

(4) $x + [(-1) \cdot x] = \mathbf{0}$ for all $x \in \mathbb{R}^n$;

(5) $(\lambda + \mu) \cdot x = (\lambda \cdot x) + (\mu \cdot x)$ for all $x \in \mathbb{R}^n$, $\lambda, \mu \in \mathbb{R}$;

(6) $(\lambda \mu) \cdot x = \lambda \cdot (\mu \cdot x)$ for all $x \in \mathbb{R}^n$, $\lambda, \mu \in \mathbb{R}$;

(7) $\lambda \cdot (x+y) = (\lambda \cdot x) + (\lambda \cdot y)$ for all $x, y \in \mathbb{R}^n$, $\lambda \in \mathbb{R}$;

(8) $1 \cdot x = x$ for all $x \in \mathbb{R}^n$.

We denote $x + [(-1) \cdot y]$ by $x - y$ for $x, y \in \mathbb{R}^n$.

$\boxed{8}$ $\mathbb{R}^n$ is a <u>metric space</u> with <u>the metric</u> $\rho : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ defined by:

$$\rho(x,y) \stackrel{\text{def}}{=\!=} \sqrt{\sum_{j=1}^{n} (x_j - y_j)^2}\,, \quad \text{for } x = (x_1, x_2, \cdots, x_n),\ y = (y_1, y_2, \cdots, y_n) \in \mathbb{R}^n.$$

ρ is a metric because it has the following properties:

(1) $\rho(x,y) \geqslant 0$ for all $x, y \in \mathbb{R}^n$;
$\rho(x,y) = 0$ iff $x = y$;

(2) $\rho(x,y) = \rho(y,x)$ for all $x, y \in \mathbb{R}^n$;

and (3) $\rho(x,z) \leqslant \rho(x,y) + \rho(y,z)$ for all $x, y, z \in \mathbb{R}^n$.

We prove (3) — the so-called triangle inequality 

— later, while (1) and (2) are rather obviously true. ρ is called <u>the usual metric</u> on $\mathbb{R}^n$, and it is induced by the <u>norm</u> $\|\ \|$: $\rho(x,y) = \|x - y\|$, for all $x, y \in \mathbb{R}^n$, where $\|z\| = \sqrt{\sum_{j=1}^{n} z_j^2}$ for $z = (z_1, z_2, \cdots, z_n) \in \mathbb{R}^n$.

The map $\|\ \| : \mathbb{R}^n \to \mathbb{R}$ given above is a norm on $\mathbb{R}^n$, because it has the following properties:

(i) $\|z\| \geqslant 0$ for every $z \in \mathbb{R}^n$; $\|z\| = 0$ iff $z = 0$;

(ii) $\|\lambda \cdot z\| = |\lambda|\ \|z\|$ for every $z \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}$;

(iii) $\|x + y\| \leqslant \|x\| + \|y\|$ for all $x, y \in \mathbb{R}^n$.

The above (i) and (ii) are obviously true; the proof of (iii) — also called the triangle inequality — will be given a bit later. The norm $\| \ \|$ is, in turn, induced by the inner product: $\|z\| = \langle z, z \rangle^{1/2}$, where

$$\langle x, y \rangle \stackrel{\text{def}}{=\!=} \sum_{j=1}^{n} x_j y_j \quad \text{for } x = (x_1, \cdots, x_n),\ y = (y_1, \cdots, y_n) \in \mathbb{R}^n.$$

The above mapping $\langle\ ,\ \rangle : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is an inner product on $\mathbb{R}^n$, because it has the following properties:

(α) $\langle x, x \rangle \geq 0$ for every $x \in \mathbb{R}^n$;
$\langle x, x \rangle = 0$ iff $x = \mathbf{0} \in \mathbb{R}^n$;

(β) $\langle \lambda \cdot x + y, z \rangle = \lambda \langle x, z \rangle + \langle y, z \rangle$ for all $x, y, z \in \mathbb{R}^n$, and $\lambda \in \mathbb{R}$;

(γ) $\langle x, y \rangle = \langle y, x \rangle$ for all $x, y \in \mathbb{R}^n$.

An arbitrary subset S of $\mathbb{R}^n$ has an (induced, or inherited) metric

$$\rho_S(x, y) \stackrel{\text{def}}{=\!=} \rho(x, y) \quad \forall x, y \in S.$$

Sometimes, for simplicity, the subscript S in ρ_S is suppressed. Likewise, any linear subspace W (i.e. $W \subset \mathbb{R}^n$ such that

$+(W\times W)\subset W$ and $\bullet(\mathbb{R}\times W)\subset W$) of $\mathbb{R}^n$ has an inherited norm and inner product.

We now give a proof of the following inequality (the so-called Cauchy-Schwartz inequality), which clearly implies (iii), hence (3), above:

For all $x, y\in\mathbb{R}^n$: $|\langle x,y\rangle| \leq \|x\|\,\|y\|$.

Proof. Observe that for all $t\in\mathbb{R}$:

$$0\leq\langle x+t\cdot y,\ x+t\cdot y\rangle = \cdots = \|x\|^2+2\langle x,y\rangle t+\|y\|^2t^2$$

(using (α), (β), (γ) above). Suppose first that $\|y\|\neq 0$; then we can put $t=-\langle x,y\rangle\|y\|^{-2}$ into the above, and obtain $0\leq\|x\|^2-|\langle x,y\rangle|^2\|y\|^{-2}$, hence $|\langle x,y\rangle|\leq\|x\|\,\|y\|$.

If $\|y\|=0$, then $y=0$, $y=0\cdot y$, $\langle x,y\rangle=\langle y,x\rangle=\langle 0\cdot y,x\rangle = 0\langle y,x\rangle=0$, and the desired inequality follows easily.

Remark The above proof works for any real inner product space.

We can consider sequences $(a_n)_{n\in\mathbb{N}}$ where each $a_n \in \mathbb{R}^p$ for a fixed p (independent of n). Such a sequence is said to be in $\mathbb{R}^p$. The concept of "$\lim\limits_{n\to\infty} a_n = \ell$" where $\ell \in \mathbb{R}^p$ is defined analogously as in the case $p=1$, using the norm $\| \ \|$ in $\mathbb{R}^p$ instead of $| \ |$ in $\mathbb{R}$. We have analogous theorems. The following ((Cauchy) convergence test is also valid:

<u>Thm</u> A sequence (a_n) in $\mathbb{R}^p$ converges to some $\ell \in \mathbb{R}^p$ iff the following condition holds:

(*) for each $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$\|a_n - a_m\| < \varepsilon \quad (\text{i.e. } \rho(a_n, a_m) < \varepsilon)$$

whenever $n, m \geq N$.

A sequence (a_n) is said to be <u>Cauchy</u> if condition (*) holds. A metric space (X, ρ) is said to be <u>complete</u> (or metrically complete) if every Cauchy sequence in X (w.r.t. ρ) converges in X (w.r.t. ρ). Thus <u>$\mathbb{R}^p$ is complete</u> (w.r.t. ρ). A normed [resp., inner product] space is said to be complete if it is complete w.r.t. the induced metric.

So <u>$\mathbb{R}^p$ is a complete inner product space</u>.