Analysis II

Chapter 0. Set Theory

For this course, we need just a few results/concepts of set theory, which are covered in Reading Material 1 uploaded on our homepage. You may consult the material as/when it is needed.

Notation

$\mathbb{R}$ = the set of (all) real numbers

$\mathbb{N}$ = the set of (all) positive integers (not including 0)

$\mathbb{Z}$ = the set of (all) integers

$\mathbb{Q}$ = the set of (all) rational numbers

$\mathbb{Q}_+$ = the set of (all) positive rational numbers (excluding 0)

ϕ = the empty set

$\exists$ = "there exist(s)", $\forall$ = "for all/every/each"

$f: A \to B: x \mapsto f(x)$

Analysis I

Chapter 1 Euclidean Spaces

$\boxed{1}$ $\mathbb{R}$ is a complete Archimedean ordered field in the sense that $(\mathbb{R}; +, \cdot; \leq)$ satisfies the following four conditions (C1) – (C4).

(C1) $(\mathbb{R}, +, \cdot)$ is a field, i.e. the operations $+$ and $\cdot$ (addition and multiplication) observe the usual algebraic rules e.g. $\alpha \cdot (\beta + \gamma) = (\alpha \cdot \beta) + (\alpha \cdot \gamma)$, each non-zero $\alpha \in \mathbb{R}$ has a multiplicative inverse $\alpha^{-1} \in \mathbb{R}$ such that $\alpha \cdot \alpha^{-1} = 1$, etc.

(C2) The field $(\mathbb{R}, +, \cdot)$ is ordered by $\leq$, i.e. the (usual) order $\leq$ is induced by the set P of all positive (real) numbers: $\alpha \leq \beta$ means $\beta - \alpha \in P \cup \{0\}$, and P satisfies:

(i) $\alpha+\beta \in P$ for all $\alpha, \beta \in P$,

(ii) $\alpha \cdot \beta \in P$ for all $\alpha, \beta \in P$,

(iii) for each $\alpha \in \mathbb{R}$, one and only one of the following holds:

$$\alpha \in P, \quad \alpha = 0, \quad -\alpha \in P$$

where $-\alpha$ denotes the additive inverse (the negative) of α in $\mathbb{R}$.

(C3) <u>The order $\leq$ is Archimedean</u>, i.e. for each $\alpha \in \mathbb{R}$, there exists $n \in \mathbb{N}$ such that $n > \alpha$.

(C4) <u>$(\mathbb{R}, \leq)$ is (order-) complete</u>, i.e. Thm 1 below holds.

A subset S of $\mathbb{R}$ is said to be <u>bounded above</u> if there exists $\gamma \in \mathbb{R}$ such that $s \leq \gamma$ for all $s \in S$. In this case, γ is called an <u>upper bound</u> of S, and S is said to be <u>bounded</u>

above by r. Similarly we define a "bounded below" subset and a "lower bound". A real number t is called a least element of S if $t \in S$ and if all $s \in S$, $t \leq s$. Such a t may, or may not, exist; in case it does exist, it is unique. Similarly we define a "greatest element" of S.

Thm 1 (Order completeness) For every bounded above, non-empty subset S of $\mathbb{R}$, the set of all upper bounds of S in $\mathbb{R}$ has a least element. (This least element is called the least upper bound, or supremum, of S, and is denoted by $\sup S$. It may, or may not, belong to S.)

Notes

(i) Thm 1 is equivalent to the following

<u>Thm 1'</u> For every bounded below, non-empty subset S of $\mathbb{R}$, the set of all lower bounds of S in $\mathbb{R}$ has a greatest element.

greatest lower bound

(This greatest element, called the <u>infimum</u> of S, is denoted by <u>$\inf S$</u>. It may, or may not, belong to S.)

(ii) $(\mathbb{Q}, +, \cdot\,; \leq)$ is also an Archimedean ordered field (where, as usual, $\mathbb{Q}$ denotes the set of all rational numbers), but it is NOT order complete. Indeed, letting $a_1 = 3$, $a_{n+1} = \frac{1}{2}(a_n + \frac{8}{a_n})$, $n \in \mathbb{N}$, $b_n = \frac{8}{a_n}$, $n \in \mathbb{N}$, and $S = \{a_n : n \in \mathbb{N}\}$, $T = \{b_n : n \in \mathbb{N}\}$, we can show that S does not admit an infimum in $\mathbb{Q}$, and T does not admit a supremum in $\mathbb{Q}$, even though they are bounded above and below in $\mathbb{Q}$.

(Check: $0 < a_n^2 - 8 < \frac{(a_{n-1}^2 - 8)^2}{32}$, $0 < a_n^2 - 8 < 32^{1-2^{n-1}}$ for all integers $n \geq 2$.
$\therefore 0 < a_n - \sqrt{8} = \frac{a_n^2 - 8}{a_n + \sqrt{8}} < 6(32^{-2^{n-1}})$ as $a_n > \sqrt{8}$, $\therefore 0 < a_n - \sqrt{8} < 10 \cdot 10^{-3 \cdot 2^{n-2}}$ as $32^2 > 10^3$.
From this, we get $0 < \sqrt{8} - b_n < 10 \cdot 10^{-3 \cdot 2^{n-2}}$ ~~too~~.)

Here we give more details of the checking/arguement.

Note: $a_n^2-8=\frac{1}{4}\left(a_{n-1}^2+16+\frac{64}{a_{n-1}^2}\right)-8$

$$=\frac{1}{4}\left(a_{n-1}^2-16+\frac{64}{a_{n-1}^2}\right)=\frac{1}{4}\frac{(a_{n-1}^2-8)^2}{a_{n-1}^2},$$

$\therefore$ (by ind.) $a_n^2-8>0$ (hence $\frac{1}{a_n^2}<\frac{1}{8}$)

and $a_n^2-8<\frac{1}{4}\cdot\frac{(a_{n-1}^2-8)^2}{8}$.

Thus $0<a_n^2-8<\frac{(a_{n-1}^2-8)^2}{32}$ for all $n\in\mathbb{N}$.

It follows: $a_2^2-8<\frac{(a_1^2-8)^2}{32}=32^{1-2^{2-1}}$,

$$a_3^2-8<32^{-1}(a_2^2-8)^2<32^{-1+2(1-2^{2-1})}=32^{1-2^{3-1}},$$

$$\vdots$$

$$a_n^2-8<32^{1-2^{n-1}}, \text{ for every } n\geq 2$$

(by induction). Therefore

$$0<a_n-\sqrt{8}=\frac{a_n^2-8}{a_n+\sqrt{8}}<\frac{a_n^2-8}{2\sqrt{8}}<\frac{32\cdot 32^{-2^{n-1}}}{\sqrt{32}}<6\cdot 32^{-2^{n-1}}, \quad n\in\mathbb{N}$$

$\therefore$ $0<a_n-\sqrt{8}<10\cdot 10^{-3\cdot 2^{n-2}}$ $\cdots\cdots$ ① (as $32^2>10^3$; actually we can just keep $6\cdot 32^{-2^{n-1}}$ throughout.)

Then, as $\sqrt{8}-b_n=\sqrt{8}-\frac{8}{a_n}=\frac{\sqrt{8}}{a_n}(a_n-\sqrt{8})$, we have also

$$0<\sqrt{8}-b_n<10\cdot 10^{-3\cdot 2^{n-2}} \quad \cdots\cdots ②$$

Thus we have: $\sqrt{8}-10^{1-3\cdot 2^{n-2}}<b_n<\sqrt{8}<a_n<\sqrt{8}+10^{1-3\cdot 2^{n-2}}$ for all $n\in\mathbb{N}$.
(from ① & ②) $\cdots\cdots$ ⊛

To see that S admits no inf in $\mathbb{Q}$, let $g\ (\in\mathbb{Q})$ be a lower bd of S in $\mathbb{Q}$, i.e. $a_n\geq g\in\mathbb{Q}$ for all $n\in\mathbb{N}$. Then from ⊛, $\sqrt{8}\geq g$. But $\sqrt{8}\notin\mathbb{Q}$, $\therefore\sqrt{8}>g$. From ⊛, $0<a_n-b_n<2\cdot 10^{1-3\cdot 2^{n-2}}$; so $\exists\, N\in\mathbb{N}$ s.t. $0<a_N-b_N<(\sqrt{8}-g)$.
It follows that $g+(a_N-b_N)\in\mathbb{Q}$, and $g<g+a_N-b_N<g+(\sqrt{8}-g)=\sqrt{8}<a_n$ for all $n\in\mathbb{N}$. Thus $g+a_N-b_N$ is a lower bd of S, which is larger than g. So S admits no inf.

(iii) Note that a real number $t = \sup S$ (in $\mathbb{R}$) iff the following two conditions are satisfied:

(α) for all $s \in S$, $s \leq t$;

(β) for each $v \in \mathbb{R}$ satisfying $v < t$, $\exists\, s_v \in S$ such that $v < s_v$.

Similarly $r = \inf S$ iff the following two conditions are satisfied:

(γ) for all $s \in S$, $r \leq s$;

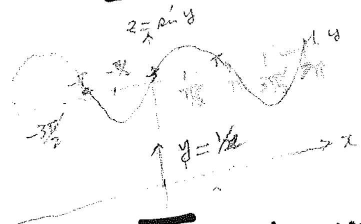
(δ) for each $v \in \mathbb{R}$ satisfying $r < v$, $\exists\, s_v \in S$ such that

$$s_v < v.$$

<u>Exercise</u>

Is the set $S = \{x \in \mathbb{R} : 0 < \sin(\frac{1}{x}) < \frac{1}{\sqrt{2}}\}$ bounded below/above in $\mathbb{R}$? Find $\sup S$ and $\inf S$ (if exists).

Ans./Sol. S is not bounded above; $\sup S$ is undefined.



S is bounded below; $\inf S = -\frac{1}{\pi}$.

To prove the last assertion, note first that for all $x \in S \cap (-\infty, 0)$, $x^{-1} < -\pi$, i.e. $x > -\frac{1}{\pi}$, so condition(γ) is satisfied with $r = -\frac{1}{\pi}$. $-\frac{1}{\pi}$ is a lower bound of S (in $\mathbb{R}$). Secondly, let $v > -\frac{1}{\pi}$. If $v \geq 0$, then condition (γ) is obviously satisfied with $r = -\frac{1}{\pi}$. Thus we assume $v < 0$. Then $\frac{1}{v} < -\pi$. Therefore there exists t_v such that $\frac{1}{v} < t_v \in (-\pi - \frac{\pi}{4}, -\pi)$. Let $s_v \overset{\text{def}}{=} \frac{1}{t_v}$. Then $\frac{1}{s_v} = t_v$, $\sin(\frac{1}{s_v}) \in (0, \frac{1}{\sqrt{2}})$ (thus $s_v \in S$) and $s_v < v$. Therefore (condition (δ) is also satisfied with $r = -\frac{1}{\pi}$)

$-\frac{1}{\pi} = \inf S$.

The arguement after "Secondly" (presented above) can be shortened as follows: Because $\exists t \in (-\pi - \frac{\pi}{4}, -\pi)$ and close to $-\frac{1}{\pi}$ (can be chosen as close as you like), it follows that $s \overset{\text{def}}{=} \frac{1}{t} > -\frac{1}{\pi}$, s (can be chosen as) close to $-\frac{1}{\pi}$ (as you like), hence we see that the condition (γ) is satisfied with $r = -\frac{1}{\pi}$.

$\boxed{2}$ $\mathbb{R}$ is a normed linear space over itself

<u>$\mathbb{R}$ is a linear space</u> over itself in the following sense:

1. for all $a, b \in \mathbb{R}$: $a+b=b+a$;
2. for all $a, b, c \in \mathbb{R}$: $(a+b)+c=a+(b+c)$;
3. for all $a \in \mathbb{R}$: $a+0=a$ (here 0 is the usual number zero);
4. for all $a \in \mathbb{R}$: $a+(-a)=0$;
5. for all $a, b \in \mathbb{R}$, $\alpha \in \mathbb{R}$: $\alpha(a+b)=\alpha a+\alpha b$;
6. for all $a \in \mathbb{R}$, $\alpha, \beta \in \mathbb{R}$: $(\alpha+\beta)a=\alpha a+\beta a$;
7. for all $a \in \mathbb{R}$, $\alpha, \beta \in \mathbb{R}$: $\alpha(\beta a)=(\alpha\beta)a$;
8. for all $a \in \mathbb{R}$: $1(a)=a$ (here 1 is the usual number one).

As a linear space over itself, <u>$\mathbb{R}$ is normed</u>; in fact, the absolute value is a <u>norm</u> on $\mathbb{R}$, i.e. $|\ |: \mathbb{R} \to [0, \infty)$: $a \mapsto |a|$ ($=a$ if $a \geq 0$, $=-a$ if $a \leq 0$) satisfies:

9. $|a+b| \leq |a|+|b|$, for all $a, b \in \mathbb{R}$,
10. $|\alpha a| = |\alpha|\,|a|$, for all $a, \alpha \in \mathbb{R}$.

This norm gives arise the following metric (on $\mathbb{R}$) $\rho: \mathbb{R}\times\mathbb{R} \to [0, \infty)$: $(a, b) \mapsto |a-b| = \rho(a, b)$.

This metric is called the <u>usual</u>/<u>standard</u> <u>metric</u> on $\mathbb{R}$; it satisfies (so <u>$\mathbb{R}$ is a metric space</u>):

(i) $\rho(a, b) \geq 0$, $=0$ iff $a=b$;

(ii) $\rho(a, b) \leq \rho(a, c)+\rho(c, b)$, for all $a, b, c \in \mathbb{R}$;

(iii) $\rho(a, b)=\rho(b, a)$, for all $a, b \in \mathbb{R}$.

$\boxed{3}$ By definition, a real sequence, or a sequence of real numbers, is a map $a : \mathbb{N} \to \mathbb{R} : n \mapsto a_n$; the sequence is usually denoted by $(a_n)_{n \in \mathbb{N}}$ or (a_n). We say that a sequence $(a_n)_{n\in\mathbb{N}}$ <u>converges</u> to a real number ℓ, in symbols $\lim_{n\to\infty} a_n = \ell$, if for each $\varepsilon > 0$, there exists $N \in \overset{\mathbb{R}}{\mathbb{N}}$ such that for all $n \geq N$,

$$|a_n - \ell| \overset{\leq}{<} \varepsilon, \quad \text{i.e.} \quad \ell - \varepsilon \overset{\leq}{<} a_n \overset{\leq}{<} \ell + \varepsilon .$$

In other words,

$$|a_n - \ell| < \varepsilon \quad \text{for sufficiently large } n.$$

In this case, ℓ is unique, and is called the <u>limit</u> of $(a_n)_{n\in\mathbb{N}}$. A sequence need not converge to any real number; if it does, the limit is unique.

A sequence (a_n) is said to be <u>bounded above</u>

[respectively, bounded below] if the set $\{a_n : n \in \mathbb{N}\}$ is bounded above [resp., bounded below] as a subset of $\mathbb{R}$. (a_n) is said to be bounded if it is bounded above and bounded below. (a_n) is increasing [resp., decreasing] if $a_n \leq a_{n+1}$ [resp., $a_n \geq a_{n+1}$] for all $n \in \mathbb{N}$.

Below are some basic properties of convergence and limits.

Thm 2

(1) If (a_n) converges to a real number, then (a_n) is bounded.

(2) If $\lim_{n\to\infty} a_n = a > b$ [resp., $a < b$], then for sufficiently large n, $a_n > b$ [resp., $a_n < b$].

(3) If $a_n \leq b_n$ [resp. $a_n \geq b_n$] for sufficiently large n, if $\lim_{n\to\infty} b_n = b$, and if $\lim_{n\to\infty} a_n = a$, then $a \leq b$ [resp., $a \geq b$].

{Note that even if $a_n < b_n$ [resp., $a_n > b_n$] for all $n \in \mathbb{N}$, it may happen that $a = b$.}

(4) (Sandwich Thm.) If $a_n \leq b_n \leq c_n$ for sufficiently large n, and if $\lim_{n\to\infty} a_n = l = \lim_{n\to\infty} c_n$, then

$$\lim_{n\to\infty} b_n = l.$$

(5) Suppose $\lim_{n\to\infty} a_n = a$, $\lim_{n\to\infty} b_n = b$ (and $a, b \in \mathbb{R}$).

Then

(i) $\lim_{n\to\infty} (a_n + b_n) = a + b$, (ii) for every $\lambda \in \mathbb{R}$, $\lim_{n\to\infty} (\lambda a_n) = \lambda a$,

(iii) $\lim_{n\to\infty} (a_n b_n) = ab$, (iv) $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{a}{b}$ if $b \neq 0$.

(6) If (a_n) is bounded above and increasing [resp., bounded below and decreasing], then

$\lim_{n\to\infty} a_n = \sup\{a_n : n \in \mathbb{N}\}$ [resp., $= \inf\{a_n : n \in \mathbb{N}\}$].

$\boxed{4}$ At this juncture, we make a small digression. For convenience, we write $\lim_{n\to\infty} a_n = \overset{+\infty}{\infty}$ if for each $r \in \mathbb{R}$ (no matter how large r is), there exists $N \in \mathbb{N}$ such that $a_n > r$ whenever $n \geq N$. Likewise, we write $\lim_{n\to\infty} a_n = -\infty$ if for each $r \in \mathbb{R}$, there exists $N \in \mathbb{N}$ such that $a_n < r$ whenever $n \geq N$. We have the following

<u>Prop. 3</u>

(1) If $\lim_{n\to\infty} a_n = 0$ and if (b_n) is bounded, then

$$\lim_{n\to\infty} (a_n b_n) = 0.$$

(2) If $a_n > 0$ [resp. $a_n < 0$] for all $n \in \mathbb{N}$ and if $\lim_{n\to\infty} a_n = 0$, then

$$\lim_{n\to\infty} a_n^{-1} = \infty \text{ [resp. } -\infty].$$

(3) If $\lim_{n\to\infty} a_n = \pm\infty$ and if $a_n \neq 0$ for all $n \in \mathbb{N}$, then

$$\lim_{n\to\infty} a_n^{-1} = 0.$$

(4) If $\lim_{n\to\infty} a_n = \pm\infty$ and if $b_n \geq b > 0$ [resp. $b_n \leq b < 0$] for sufficiently large n, then $\lim_{n\to\infty} a_n b_n = \pm\infty$ [resp $\mp\infty$].