# Ch8 Morse index form and Bonnet-Myers Theorem

Let $\gamma$ = normalized geodesic defined on $[a,b]$

$$\mathcal{V} = \mathcal{V}(a,b) = \left\{ \begin{array}{l} X = \text{piecewise } C^\infty \text{ vector field along } \gamma \text{ s.t.} \\ \langle X, \dot\gamma \rangle = 0 \end{array} \right\}$$

$$\mathcal{V}_0 = \mathcal{V}_0(a,b) = \{ X \in \mathcal{V} : X(a) = X(b) = 0 \}$$

Note $\mathcal{V}_0$ = space of transversal vector fields of <u>normal</u> variations of $\gamma$.

**Def**: (1) $$I(X,X) = \int_a^b \left[ |\dot X(t)|^2 - \langle R_{\dot\gamma X} \dot\gamma, X \rangle \right] dt, \quad \forall X \in \mathcal{V}$$

(where $\dot X(t) = D_{\dot\gamma} X(t)$ )

( Note: $\int_a^b |\dot{X}(t)|^2 \overset{\text{def}}{=} \sum_{i=0}^{k-1} \int_{a_i}^{a_{i+1}} |\dot{X}(t)|^2 dt$

where $a = a_0 < a_1 < \dots < a_k = b$ s.t. $X\big|_{[a_i, a_{i+1}]} \in C^\infty$ )

(2) $I(X,Y) \overset{\text{def}}{=\!=} \frac{1}{2}\Big[ I(X+Y, X+Y) - I(X,X) - I(Y,Y) \Big]$

$\forall X, Y \in \mathcal{V}$,

is called the <u>index form</u> of $\gamma$.

<u>Notes</u>: (i) $I(X,Y) = \int_a^b \big[ \langle \dot{X}, \dot{Y} \rangle - \langle R_{\dot{\gamma} X} \dot{\gamma}, Y \rangle \big](t)\, dt$

(Ex!)

(ii) $I(X,Y)$ is bilinear (symmetric) (Ex)

(iii) If $U$ = transversal vector field of a normal variation $\{\gamma_u\}$ of the normalized geodesic $\gamma$, then $U \in \mathcal{V}_0 \, (\subset \mathcal{V})$

and the 2nd variation

$$L''(0) = I(U, U) \quad \left(\text{by 2nd variation formula}\right)$$

<u>Lemma 1</u>: Let
- $\gamma : [a,b] \to M$ normalized geodesic
- $\gamma(b)$ conjugate to $\gamma(a)$

Then $\forall$ normal Jacobi field $U$ with $U(a) = U(b) = 0$ satisfies

$$I(U, U) = 0$$

Pf:

$$I(U,U) = \int_a^b \left[ |\dot{U}|^2 - \langle R_{\dot{\gamma} U} \dot{\gamma}, U \rangle \right]$$

$$= \int_a^b \left[ |\dot{U}|^2 + \langle \ddot{U}, U \rangle \right] \quad \left( U \text{ is Jacobi} \right)$$

$$= \int_a^b \left[ |\dot{U}|^2 + \langle \dot{U}, U \rangle' - |\dot{U}|^2 \right]$$

$$= \langle \dot{U}, U \rangle \Big|_a^b = 0 \quad \#$$

Note: Therefore, if $\gamma(b)$ conjugate to $\gamma(a)$, then the index form of $\gamma$ is degenerate.

Terminology: A geodesic $\gamma : [a,b] \to M$ is said to contain no conjugate point if $\gamma(a)$ has no conjugate point along $\gamma$.

Lemma 2 Let
- $\gamma : [a,b] \to M$ normalized geodesic
- $\gamma$ has no conjugate point

Then $I(X, X)$ is positive definite on $\mathcal{V}_0(a,b)$.

<u>Lemma 3</u> Let
- $\gamma : [a,b] \to M$ normalized geodesic
- $\gamma(b)$ conjugate to $\gamma(a)$
- $\gamma(c)$ is not conjugate to $\gamma(a)$ for $c \in (a,b)$

Then $I(X,Y)$ is <u>semi-positive</u> <u>definite</u> on $\mathcal{V}_0(a,b)$ but <u>not</u> positive definite.

<u>Lemma 4</u> Let $\gamma : [a,b] \to M$ normalized geodesic

Then $\exists\ c \in (a,b)$ s.t. $\gamma(c)$ is conjugate to $\gamma(a)$

$\iff \exists\ X \in \mathcal{V}_0(a,b)$ s.t. $I(X,X) < 0$.

<u>Cor</u>: If $\gamma : [a,b] \to M$ is a normalized geodesic which contains no conjugate point, then $\forall\ [\alpha,\beta] \subset [a,b]$, $\gamma|_{[\alpha,\beta]}$ has no conjugate point.

Pf: Suppose not, then $\exists$ $[\alpha,\beta]$ s.t. $\gamma(\beta)$ conjugate to $\gamma(\alpha)$

Then by Lemma 3, $\exists$ $J \not\equiv 0 \in \mathcal{V}_0(\alpha,\beta)$ s.t.

$$I(J,J)=0 \qquad (J(\alpha)=J(\beta)=0)$$

Define a piecewise $C^\infty$ vector field $X$ along $\gamma:[a,b]\to M$

by $$X=\begin{cases} J, & t\in[\alpha,\beta] \\ 0, & \text{otherwise} \end{cases}$$

Then $X$ is well-defined & $\in \mathcal{V}_0(a,b)$.

$$I(X,X)=I_a^b(X,X)=\int_a^b\left[|\dot{X}|^2-\langle R_{\dot\gamma X}\dot\gamma, X\rangle\right]$$

$$=\int_\alpha^\beta\left[|\dot{J}|^2-\langle R_{\dot\gamma J}\dot\gamma, J\rangle\right]=I_\alpha^\beta(J,J)=0.$$

Hence Lemma 2 $\Rightarrow$ $\gamma:[a,b]\to M$ contains conjugate point,

contradiction $\#$

Claim: For $X, Y \in C^\infty$

$$(*) \quad I(X,Y) = \langle \dot{X}, Y \rangle \Big|_a^b - \int_a^b \langle \ddot{X} + R_{\dot\gamma X}\dot\gamma, Y \rangle (t)\, dt$$

Pf: $$I(X,Y) = \int_a^b \left[ \langle \dot{X}, \dot{Y} \rangle - \langle R_{\dot\gamma X}\dot\gamma, Y \rangle \right]$$

$$= \int_a^b \left[ \langle \dot{X}, Y \rangle' - \langle \ddot{X}, Y \rangle - \langle R_{\dot\gamma X}\dot\gamma, Y \rangle \right]$$

$$= \langle \dot{X}, Y \rangle \Big|_a^b - \int_a^b \langle \ddot{X} + R_{\dot\gamma X}\dot\gamma, Y \rangle \quad \#$$

Claim: For piecewise $C^\infty$ $X, Y$ s.t.

$X \in C^\infty[a_i, a_{i+1}]$ where $a = a_0 < a_1 < \cdots < a_k = b$,

$$I(X,Y) = \sum_{i=0}^{k-1} \langle \dot{X}_i, Y \rangle \Big|_{a_i}^{a_{i+1}} - \sum_{i=0}^{k-1} \int_{a_i}^{a_{i+1}} \langle \ddot{X}_i + R_{\dot\gamma X_i}\dot\gamma, Y \rangle \, dt$$

where $X_i = X|_{[a_i, a_{i+1}]}$, $i = 0, \cdots, k-1$

Lemma5 : Let • $\gamma : [a,b] \to M$ normalized geodesic

• $V \in \mathcal{V}(a,b)$

Then $I(V, \mathcal{V}_0) = 0 \iff V$ is a Jacobi field.

Pf : ($\Leftarrow$) By (*)

$$I(V,Y) = \underbrace{\sum_{i=0}^{k-1} \langle \dot{V}, Y \rangle \Big|_{a_i}^{a_{i+1}}}_{\substack{= \\ 0}} - \underbrace{\sum_{i=0}^{k-1} \int_{a_i}^{a_{i+1}} \langle \ddot{V} + R_{\dot\gamma V} \dot\gamma, Y \rangle}_{\substack{= \\ 0}}$$

$\left(\begin{array}{l}\text{Jacobi} \in C^\infty \\ \& \ Y(a) = Y(b) = 0\end{array}\right)$ $\quad$ ($V$ = Jacobi)

$= 0$.

($\Rightarrow$) Suppose $I(V, \mathcal{V}_0) = 0$

Since $V$ is piecewise $C^\infty$, $\exists\ a = a_0 < a_1 < \cdots < a_k = b$

s.t. $V_i := V|_{[a_i, a_{i+1}]} \in C^\infty$, $i = 0, \cdots, k-1$.

Take a $C^\infty$ function $f$ on $[a,b]$ s.t.

$$\begin{cases} f(a_i) = 0, & \forall\ i = 0, \cdots, k-1 \\ f > 0 & \text{otherwise} \end{cases}$$

Let $X = V$, $Y = f(\ddot{V} + R_{\dot{\gamma}V}\dot{\gamma})$

Then $Y$ is well-defined & $\in \mathcal{V}_0$

Hence $(*) \Rightarrow$

$$0 = I(V, Y) = \sum_{i=0}^{k-1} \langle \dot{V}_i, Y \rangle \Big|_{a_i}^{a_{i+1}}$$

$$-\sum_{i=0}^{k-1}\int_{a_i}^{a_{i+1}}\langle \ddot{U}+R_{\dot\gamma U}\dot\gamma, f(\ddot{U}+R_{\dot\gamma U}\dot\gamma)\rangle$$

$$=-\sum_{i=0}^{k-1}\int_{a_i}^{a_{i+1}} f\,|\ddot{U}+R_{\dot\gamma U}\dot\gamma|^2 \quad \left(\text{since } Y(a_i)=0 \atop \forall i\right)$$

$$\Rightarrow \quad \ddot{U}+R_{\dot\gamma U}\dot\gamma=0 \quad \text{on } [a_i, a_{i+1}], \ \forall\ i=0,\dots,k-1$$

Putting it back to the formula $(*)$, one has

$$0=I(U,\tilde{Y})=\sum_{i=0}^{k-1}\langle \dot{U},\tilde{Y}\rangle\Big|_{a_i}^{a_{i+1}} \qquad \forall\ \tilde{Y}\in\mathcal{V}_0$$

For a fixed $i_0\in\{1,\dots,k-1\}$,

take $\tilde{Y}_{i_0}\in\mathcal{V}_0$ s.t.
$$\begin{cases}\tilde{Y}_{i_0}(a_i)=0, & \forall\ i\neq i_0\\ \tilde{Y}_{i_0}(a_{i_0})=\dot{U}_{i_0+1}(a_{i_0})-\dot{U}_{i_0}(a_{i_0})\end{cases}$$

Then $0 = I(V, \tilde{Y}_{i_0}) = -\langle \mathring{V}_{i_0+1}(a_{i_0}), \tilde{Y}_{i_0}(a_{i_0}) \rangle + \langle \mathring{V}_{i_0}(a_{i_0}), \tilde{Y}_{i_0}(a_{i_0}) \rangle$

$$= -\langle \mathring{V}_{i_0+1}(a_{i_0}) - \mathring{V}_{i_0}(a_{i_0}), \tilde{Y}_{i_0}(a_{i_0}) \rangle = -|\tilde{Y}_{i_0}(a_{i_0})|^2$$

$$\Rightarrow \quad \mathring{V}_{i_0+1}(a_{i_0}) = \mathring{V}_{i_0}(a_{i_0})$$

Since $i_0 \in \{1, \cdots, k-1\}$ is arbitrary, $V$ is in fact $C^1$.

Then uniqueness & existence theorem $\Rightarrow$ $V$ is Jacobi #

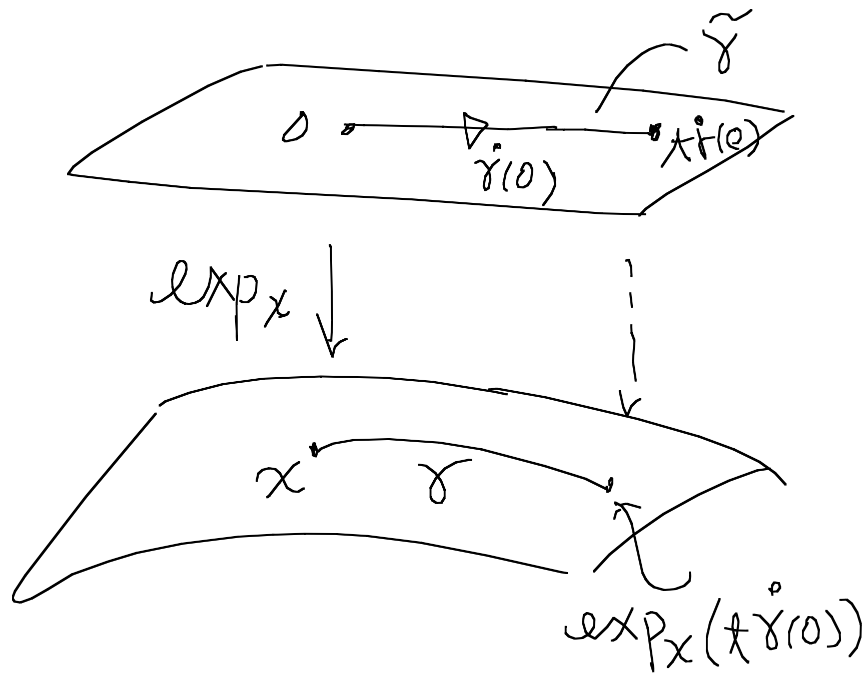
Proof of Lemma 2 :

We may assume $a = 0$, ie $\gamma : [0, b] \to M$

Define $\tilde{\gamma} : [0, b] \to T_x M$ where $x = \gamma(0)$ $\left( |\mathring{\gamma}(0)| = 1 \right)$

$$t \mapsto t\mathring{\gamma}(0)$$

By assumption, $\gamma$ has no conjugate point, and hence




$d\exp_x$ has no singular point along $\tilde{\gamma}$.

$\Rightarrow \exists$ nbd $\mathcal{U}$ of $\tilde{\gamma}([0,b])$ in $T_xM$ s.t.

$$\exp_x : \mathcal{U} \to M \text{ is a immersion.}$$

Then same proof as in Thm2 of Ch4, one can show that

(**) $\Big\{$ For any piecewise $C^\infty$ curve $\sigma : [0,b] \to \exp_x \mathcal{U}$ connecting $x$ to $\gamma(b)$, $L(\sigma) \geq L(\gamma)$. And equality holds $\Leftrightarrow$ $\sigma =$ monotonic reparametrization of $\gamma$.

Now for any variation $\{\gamma_u\}$, $u\in(-\varepsilon,\varepsilon)$. With $\varepsilon>0$ small enough, we may assume $\gamma_u \subset \exp_x \mathcal{U}$. Then by $(**)$

$$L(u) \geq L(0).$$

Since $L(u)$ is $C^\infty$, $$L''(0) = \lim_{s\to 0} \frac{L(-s)+L(s)-2L(0)}{s^2} \geq 0$$

Noting that any $X\in \mathcal{V}_0$ is a transversal vector field of a normal variation of $\gamma$, therefore $I(X,X) = L''(0) \geq 0$ $\forall X \in \mathcal{V}_0$.

Suppose that $I(X,X)=0$, we have $\forall \varepsilon>0$, $Y\in \mathcal{V}_0$

$$0 \leq I(X\pm\varepsilon Y, X\pm\varepsilon Y) = I(X,X) \pm 2\varepsilon I(X,Y) + \varepsilon^2 I(Y,Y)$$
$$= \pm 2\varepsilon I(X,Y) + \varepsilon^2 I(Y,Y)$$

$\Rightarrow \quad -\varepsilon I(Y,Y) \leq 2I(X,Y) \leq \varepsilon I(Y,Y) \quad , \forall \varepsilon > 0, Y \in \mathcal{V}_0$

Letting $\varepsilon \to 0$, we have $I(X,Y) = 0$, $\forall Y \in \mathcal{V}_0$.

Lemma 5 $\Rightarrow$ $X$ = Jacobi.

But $X(0) = X(b) = 0$ and $\gamma(b)$ is not conjugate to $\gamma(0)$.

$X \equiv 0$.

$\therefore$ $I$ is positive definite ※

<u>Lemma 6</u> (Cor to Lemma 2) (minimality of Jacobi fields)

Suppose • $\gamma : [a,b] \to M$ normalized geodesic

• $\gamma$ has no conjugate point.

• $V$ = Jacobi field along $\gamma$.

Then $\forall\, X \in \mathcal{V}(a,b)$ with $X(a)=U(a)$ & $X(b)=U(b)$,

$$I(U,U) \le I(X,X).$$

Equality holds $\Leftrightarrow$ $X = U$.

Pf: Note $U - X \in \mathcal{V}_0(a,b)$

Lemma 2 $\Rightarrow$ $0 \le I(U-X, U-X)$

$$= I(U,U) - 2I(U,X) + I(X,X)$$

$$I(U,U) = \langle \dot{U}, U\rangle\Big|_a^b - \int_a^b \langle \cancel{\ddot{U} + R_{\dot\gamma U}\dot\gamma}^{\,0}, U\rangle$$

$$= \langle \dot{U}, U\rangle\Big|_a^b$$

$$I(U,X) = \langle \dot{U}, X\rangle\Big|_a^b - \int_a^b \langle \cancel{\ddot{U} + R_{\dot\gamma U}\dot\gamma}^{\,0}, X\rangle$$

$$= \langle \dot{V}, X \rangle \Big|_a^b = \langle \dot{V}, V \rangle \Big|_a^b = I(V,V)$$

$$\therefore \quad 0 \leq I(V,V) - 2I(V,V) + I(X,X)$$

$$\Rightarrow \quad I(V,V) \leq I(X,X)$$

Equality $\Leftrightarrow$ $0 = I(V-X, V-X) \Leftrightarrow V = X$ ※

(Note : In fact Lemma 2 $\Leftrightarrow$ Lemma 6 )

<u>Proof of lemma 3</u>

It is clear that $I(X,Y)$ is not positive definite

( By Lemma 1 )

Take a parallel frame field $\{E_1(t), \cdots, E_n(t)\}$ along $\gamma$

s.t. $E_1(t) = \dot{\gamma}(t)$

Then $\forall\ X \in \mathcal{V}_0(a,b)$ $(a=0)$

$$X(t) = \sum_{i=2}^{n} f_i(t) E_i(t) \quad \text{with } f_i(0) = f_i(b) = 0$$

$\forall\ \beta \in [0,b]$, define $\tau(X) \in \mathcal{V}_0(0,\beta)$ by

$$\tau(X)(t) = \sum_{i=2}^{n} f_i\left(\frac{b}{\beta}t\right) E_i\left(\frac{b}{\beta}t\right)$$

Then

$$I_0^\beta(\tau(X), \tau(X)) = \sum_{i,j=2}^{n} f_i\left(\frac{b}{\beta}t\right) f_j\left(\frac{b}{\beta}t\right) I_0^\beta\left(E_i\left(\frac{b}{\beta}t\right), E_j\left(\frac{b}{\beta}t\right)\right)$$

$$= -\sum_{i,j=2}^{n} f_i\left(\frac{b}{\beta}t\right) f_j\left(\frac{b}{\beta}t\right) \int_0^\beta \left\langle R_{\dot{\gamma}(t) E_i\left(\frac{b}{\beta}t\right)} \dot{\gamma}(t), E_j\left(\frac{b}{\beta}t\right) \right\rangle$$

So $\lim_{\beta \to b} I_0^\beta(\tau(X), \tau(X)) = I(X, X)$.

Since $\gamma(b)$ is the unique conjugate point, Lemma 2 $\Rightarrow$ $I_0^\beta(\tau(X), \tau(X)) \geq 0$

Hence $I(X, X) \geq 0$, ie. $I$ is semi-positive definite. $\#$

To prove Lemma 4, we need

<u>Lemma 7</u> Let • $\gamma : [0, b] \to M$ normalized geodesic

• $\gamma(b)$ is not conjugate to $\gamma(0)$.

Then $\forall\, v \in T_{\gamma(b)}M$, $\exists !$ Jacobi field $V$ along $\gamma$

s.t. $V(0) = 0$ & $V(b) = v$.

( Pf : Ex! )

Proof of Lemma 4

($\Rightarrow$) If $\exists\, c \in (a,b)$ s.t. $\gamma(c)$ conjugate to $\gamma(a)$.

Then $\exists$ non-trivial normal Jacobi field $J_1$ along $\gamma$ s.t.

$$J_1(a) = J_1(c) = 0.$$

Define $J \in \mathcal{V}_0(a,b)$ by

$$J = \begin{cases} J_1, & t \in [a,c] \\ 0, & t \in [c,b] \end{cases}$$

Then $I_a^b(J,J) = I_a^c(J_1,J_1) + I_c^b(0,0) = 0$

Now take $\delta > 0$ small s.t.

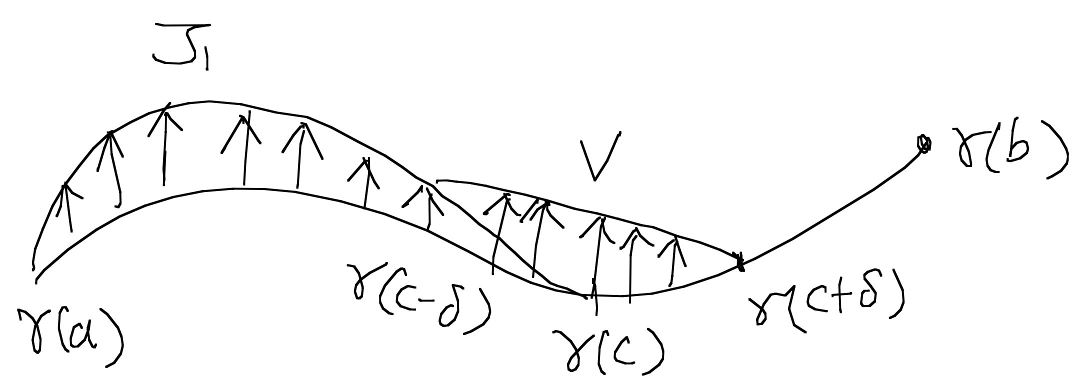
$$\exp_{\gamma(c+\delta)} : T_{\gamma(c+\delta)} M \to M$$

is diffeo. on $B(3\delta) \subset T_{\gamma(c+\delta)} M$ (and $c + \delta < b$)

Since $d(\gamma(c-\delta), \gamma(c+\delta)) < 2\delta$, $\gamma(c-\delta)$ is not conjugate to $\gamma(c+\delta)$

Then Lemma 7 $\Rightarrow$ $\exists !$ Jacobi field $V$ s.t.

$$V(c+\delta) = 0 \;\; \& \;\; V(c-\delta) = J(c-\delta) \;\; \left( = J_1(c-\delta) \right)$$



Define $U = \begin{cases} J_1, & t \in [a, c-\delta] \\ V, & t \in [c-\delta, c+\delta] \\ 0, & t \in [c+\delta, b] \end{cases}$

Then $I_a^b(U,U) = I_a^{c-\delta}(J_1, J_1) + I_{c-\delta}^{c+\delta}(V,V) + I_{c+\delta}^{b}(0,0)$

$$\left( \underset{I_{c-\delta}^{c+\delta}(J,J)}{\wedge} \quad \text{(by Lemma 6)} \right)$$

$$< I_a^{c-\delta}(J,J) + I_{c-\delta}^{c+\delta}(J,J) + I_{c+\delta}^{b}(J,J)$$

$$= I_a^b(J,J) = 0$$

$\#$

($\Longrightarrow$) If $\exists\, U \in \mathcal{V}_0(a,b)$ s.t. $I(U,U) < 0$, then Lemmas 2 & 3

$\Rightarrow \exists$ conjugate point to $\gamma(a)$ in $\gamma([a,b])$ $\#$

<u>Fact</u> (Ex.) Applying Lemma 4 to $S^2$, show that if $b > \pi$,

(***) then $\exists$ a piecewise smooth $f_0 : [0,b] \to \mathbb{R}$ s.t.

$$\begin{cases} f_0(0) = f_0(b) = 0 \\ \int_0^b \left( (f_0')^2 - f_0^2 \right) < 0 \end{cases}$$

<u>Thm 8</u> (Bonnet-Myers)

Let
- $M$ = complete Riem mfd
- $\mathrm{Ricci}_M \geq (n-1)c$, $c > 0$

Then $M$ is compact and $\mathrm{diam}(M) \leq \pi/\sqrt{c}$.

Pf: Scaling $\Rightarrow$ we may assume $C=1$.

Then we need to show if $\gamma:[0,b]\to M$ normalized <u>shortest</u> geodesic connecting $x$ to $y$, then $b\le\pi$.

Take $\underbrace{\{E_1(t),\cdots,E_n(t)\}}_{\text{parallel frame}}$ along $\gamma$ s.t. $E_1(t)=\dot\gamma(t)$

If $b>\pi$, define, for $i=2,\cdots,n$

$$X_i(t)=f_0(t)\,E_i(t)$$

where $f_0(t)$ is the function in $(***)$

Then $X_i\in\mathcal{V}_0(0,b)\ \ \forall\, i=2,\cdots,n$

& $\sum_{i=2}^{n} I(X_i, X_i) = \sum_{i=2}^{n} \int_0^b |\dot{X}_i|^2 - \langle R_{\dot\gamma X_i}\dot\gamma, X_i\rangle$

$= (n-1)\int_0^b (f_0')^2 - \int_0^b f_0^2 \sum_{i=2}^{n} \langle R_{E_1 E_i} E_1, E_i\rangle$

$\leq (n-1)\left(\int_0^b (f_0')^2 - f_0^2\right) < 0$

$\Rightarrow$ $\exists\, i_0$ s.t. $I(X_{i_0}, X_{i_0}) < 0$

$\Rightarrow$ $\gamma$ is not minimizing. Contradiction! $\therefore b \leq \pi$ #