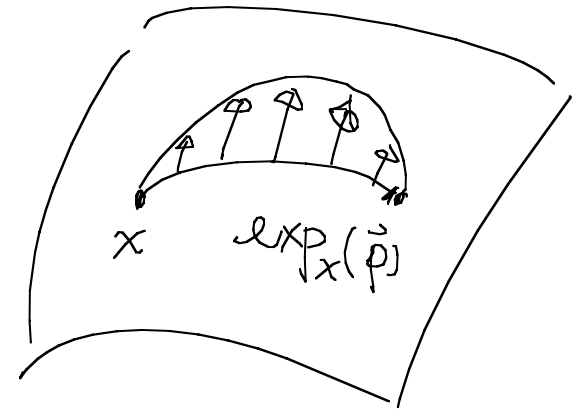## §6.2 Cartan-Hadamard Thm

**Lemma 6** $(d \exp_x)_{\vec{p}}$ is singular

$\Longleftrightarrow$ $\exists$ normal Jacobi field $U(t)$ along

$$\gamma(t) = \exp_x(t\vec{p})$$

not identically zero s.t.

$$U(0) = U(1) = 0 .$$



$x \qquad \exp_x(\vec{p})$

Pf: By the lemma right before the (original version)
Gauss lemma in ch 4,

$\left( d\exp_x \right)_{\vec{p}}$ is _non-degenerate_ in the direction of $\vec{p}$.

Therefore, we only need to consider $X$ s.t. $\langle X, \vec{p} \rangle = 0$.

Let $X \in T_x M \cong T_{\vec{p}}(T_x M)$ s.t. $\langle X, \vec{p} \rangle = 0$.

Then $\gamma_u(t) = \exp_x \left[ t(\vec{p} + u X) \right]$

gives a _normal_ Jacobi field ( by (C) )

with $U(0) = 0$, $U'(0) = X$.

Furthermore, $U(1) = (d\exp_x)_{\vec{p}}(X)$,

Therefore, if $X \in \ker \left( (d\exp_x)_{\vec{p}} \right)$ & $X \neq 0$,

then $U(t)$ is the required normal Jacobi field.

This proves the direction "$(\Longrightarrow)$".

Conversely, any normal Jacobi field is the transversal vector field of a 1-para family of geodesics given by

$$\gamma_u(t) = \exp_{S(u)} \left[ t \left( T(u) + u \, W(u) \right) \right]$$

with $S(0) = \gamma(0), \quad S'(0) = U(0)$

$T, W$ = parallel vector fields along $S(u)$.

Since $U(0) = 0$, we may take $S(u) \equiv \gamma(0) = x$

$T = \vec{p}$ & $W = \vec{z} = U'(0) \neq 0, \quad \langle T, W \rangle = 0$.

Therefore,

$$0 = U(1) = (d\exp_x)_{\vec{p}}(X)$$

$$\Rightarrow 0 \neq X \in \ker((d\exp_x)_{\vec{p}})$$

$$\therefore (d\exp_x)_{\vec{p}} \text{ is singular} \quad \text{⨯⨯}$$

Def: If $(d\exp_x)_{\vec{p}}$ is singular, then $\hat{p}$ is called

a _conjugate point_ of the map $\exp_x$,

and $\exp_x(\vec{p})$ is called a _conjugate point of $x$_

along the _geodesic_ $\gamma(t) = \exp_x(t\vec{p})$.

# Thm 7 (Cartan-Hadamard)

(1) Let $M$ be a complete Riemannian mfd. with nonpositive sectional curvature. Then $\forall x \in M$, $\exp_x : T_x M \to M$ has no conjugate point;

(2) If $M$ is a simply-connected complete Riem. mfd. s.t. for some $x \in M$, $\exp_x : T_x M \to M$ has no conjugate point, then $\exp_x : T_x M \to M$ is a diffeomorphism.

Pf of (1): Let $U$ be a normal Jacobi field with

$$U(0) = 0 \quad \text{along a geodesic } \gamma : [0, \infty) \to M.$$

Let $f(t) = \langle U(t), U(t) \rangle$ along $\gamma$, then

$$f' = 2 \langle U', U \rangle$$

$$\Rightarrow \quad f'' = 2 \langle U', U' \rangle + 2 \langle U'', U \rangle$$

$$= 2|U'|^2 - 2 \langle R_{\gamma' U} \gamma', U \rangle$$

(Jacobi eqt.)

Since $\langle R_{\gamma' U} \gamma', U \rangle = K(\text{span}(\gamma', U)) |\gamma' \wedge U|^2$

$$\leq 0 \; ;$$

we have $f'' \geq 0$,

Now suppose $\gamma(t_0)$ is a conjugate point of $x$ along some geodesic $\gamma : [0, \infty) \to M$. Then

Lemma 6 $\Rightarrow$ $\exists$ nontrivial normal Jacobi field $U(t)$ along $\gamma$ s.t. $U(0) = U(t_0) = 0$ and $U(t) \neq 0$ on $[0, t_0]$.

Applying the above, $|U(t)|^2$ is convex in $t$

$\Rightarrow$ $0 < |U(t)|^2 \leq \max\left(|U(0)|^2, |U(t_0)|^2\right) = 0$
$$\forall t \in [0, t_0]$$

Contradiction! $\#$

To prove (2), we need the following Lemmas:

Lemma 8 : Let $\varphi : M \to N$ be a local isometry between (connected) Riemannian manifolds M & N. If M is complete, then N is complete and $\varphi$ is a covering map.

Pf : Step 1 : $\varphi$ is surjective

- "$\varphi$ = local isometry" $\Rightarrow$ $\varphi(M)$ open in N.

- Suppose $\gamma \subset N$ is a geodesic such that $\gamma \cap \varphi(M) \neq \emptyset$. Then $\exists \; x \in M$ such that

$\varphi(x)$ is a point on $\gamma$. Since $\varphi$ is a local isometry, then near the point $x$, $\varphi^{-1} \circ \gamma$ defines a geodesic segment in a nbd. of $x$ in $M$ (passing thro. the point $x$). Then the completeness of $M \Rightarrow \varphi^{-1} \circ \gamma$ extends to a $\tilde{\gamma} \subset M$ defined on $(-\infty, \infty)$. By assumption on $\varphi$, we have $\varphi \circ \tilde{\gamma} : (-\infty, \infty) \longrightarrow \varphi(M) \subset N$ is a geodesic on $N$ passing thro. $\varphi(x)$ and

$$\varphi \circ \tilde{\gamma} = \varphi \circ (\varphi^{-1} \circ \gamma) = \gamma \text{ in a nbd of } 0 \in (-\infty, \infty)$$
$$(\gamma(0) = x)$$

Therefore, uniqueness of geodesic $\implies$

$$\varphi_0 \tilde{\gamma} = \gamma$$

$$\implies \gamma \subset \varphi(M).$$

So we have proved that for a geodesic segment $\gamma$ in $N$ s.t. $\gamma \cap \varphi(M) \neq \emptyset$, then $\gamma \subset \varphi(M)$. Now suppose $y$ is a limiting point of $\varphi(M)$ in $N$, then $\exists x \in M$ s.t. $\exists$ a geodesic $\gamma(t)$, $t \in [0,1]$, in $N$ such that $\gamma(0) = \varphi(x)$ and $\gamma(1) = y$.

Therefore by the above argument, $y = \gamma(1) \subset \varphi(M)$.

$\therefore \quad \varphi(M)$ is closed in $N$.

Hence $\varphi(M)$ is both open & closed ( non-empty )

in a connected manifold $N$, we have

$$\varphi(M) = N.$$

$$\Rightarrow \quad \varphi \text{ is surjective.}$$

Note : • In fact, we've proved the following commutative

diagram :

$$
\begin{array}{ccc}
T_x M & \xrightarrow{\ d\varphi\ } & T_{\varphi(x)} N \\
\exp_x^M \downarrow & & \downarrow \exp_{\varphi(x)}^N \\
M & \xrightarrow{\ \varphi\ } & N \ (\text{local isom}) \ ;
\end{array}
$$

- and N is complete,

- Even more, (for $\delta > 0$ small s.t. exp is a diffeo when restricted to a ball of radius $< \delta$.)

we have

$$B^M(\delta) \xrightarrow{\;d\varphi\;} B^N(\delta)$$

$$\exp^M_x \downarrow \qquad \overset{?}{\varphi} \qquad \downarrow \exp^N_{\varphi(y)}$$

$$B^M_\delta(x) \xrightarrow{\;\varphi\;} B^N_\delta(\varphi(y))$$

Step 2: $\varphi$ is a covering map.

i.e. We need to show that $\forall y \in N$, $\exists$ a nbd $U$ of

$y$ in $N$ such that $\varphi^{-1}(U) = \bigcup_i W_i$ with

$$
\begin{cases}
\bullet \quad W_i \cap W_j = \emptyset \quad \text{for } i \neq j \\[2mm]
\bullet \quad \varphi : W_i \to U \text{ is a diffeomorphism } \forall i.
\end{cases}
$$

**Pf of Step2:**

For any $y \in N$, $\exists \delta > 0$ such that

$$
\exp_y^N : B^N(\delta) \longrightarrow B_\delta^N \quad \text{is a diffeomorphism}
$$

where
$$
B^N(\delta) = \{ v \in T_y N : |v|_N < \delta \}
$$

$$
B_\delta^N = \{ z \in N : d_N(z, y) < \delta \}.
$$

Since $\varphi$ is a local isom & hence a local diffeo, $\varphi^{-1}(y)$ is a discrete set in $M$. Let

$$\varphi^{-1}(y) = \{x_i\}_{i \in \Lambda} \quad \text{for some index set } \Lambda.$$

and denote

$$B^i(\delta) = B^M(x_i, \delta) = \{v \in T_{x_i}M : |v|_M < \delta\}$$

$$B^i_\delta = B^M_\delta(x_i) = \{\zeta \in M : d_M(\zeta, x_i) < \delta\}.$$

Claim: (i) $\varphi^{-1}(B^N_\delta) = \bigcup_i B^i_\delta$

(ii) $\forall i : \varphi : B^i_\delta \longrightarrow B^N_\delta$ is a diffeo.

(iii) $\forall i \neq j, \quad B^i_\delta \cap B^j_\delta = \emptyset.$

Pf of (i) : It is clear that $\bigcup_i B_\delta^i \subset \varphi^{-1}(B_\delta^N)$

since $\varphi$ is a local isometry.

Conversely, for $\varphi^{-1}(B_\delta^N) \subset \bigcup_i B_\delta^i$, we take a

$z \in \varphi^{-1}(B_\delta^N)$. Then $\varphi(z) \in B_\delta^N$. By construction

of $B_\delta^N$, $\exists$ unique geodesic $\gamma : [0,1] \to B_\delta^N$

s.t. $\gamma(0) = \varphi(z)$ & $\gamma(1) = y$.

Then by the above argument (in proof of step 1),

$\exists$ a geodesic $\tilde{\gamma} : [0,1] \to M$ s.t.

$$\tilde{\gamma}(0) = z \quad \& \quad \varphi(\tilde{\gamma}(t)) = \gamma(t), \quad \forall t$$

$$\Rightarrow \quad \varphi(\tilde{\gamma}(1)) = \gamma(1) = y$$

$$\Rightarrow \quad \tilde{\gamma}(1) \in \varphi^{-1}(y) = \{x_i\}_{i \in \Lambda}.$$

$$\Rightarrow \quad \tilde{\gamma}(1) = x_i \quad \text{for some } i \in \Lambda.$$

Again, $\varphi = $ local isom $\Rightarrow$

$$\text{Length}_M(\tilde{\gamma}) = \text{Length}_N(\gamma) < \delta$$

$$\Rightarrow \quad \tilde{\gamma}(0) = z \text{ has a distance} < \delta \text{ to } x_i$$

i.e. $\quad z \in B_\delta^i \subset \bigcup_i B_\delta^i.$

This completes the proof of (i).

Pf of (ii) : By the note in step 1, & $\varphi$ is a local

isom., we have

$$B^{\tilde{i}}(\delta) \xrightarrow{d\varphi} B^N(\delta)$$

$$\exp^M_{x_i} \downarrow \quad \circlearrowleft \quad \downarrow \exp^N_y$$

$$B^{\tilde{i}}_\delta \xrightarrow{\varphi} B^N_\delta$$

i.e. $\qquad \varphi \circ \exp^M_{x_i} = \exp^N_y \circ d\varphi$

By the choice of $\delta > 0$, $\exp^N_y$ and $d\varphi$ are

diffeomorphisms. Hence $\exp^M_{x_i}$ has to be an

immersion. On the other hand $\exp^M_{x_i} : B^{\tilde{i}}(\delta) \to B^{\tilde{i}}_\delta$

is surjective ( by completeness of $M$ ), therefore we have

$$\varphi = \exp_y^N \circ d\varphi \circ (\exp_{x_i}^M)^{-1}$$

which is a diffeomorphism. This proves (ii).

Pf of (iii): Let $i \neq j \in \Lambda$. Suppose that $B_\delta^i \cap B_\delta^j \neq \emptyset$.

Then $\exists \, \zeta \in B_\delta^i \cap B_\delta^j$. Using (ii), $\exists$ geodesics

$$\tilde{\gamma}_i \in B_\delta^i \quad \& \quad \tilde{\gamma}_j \subset B_\delta^j \quad \text{joining} \ \zeta \ \text{to} \ x_i \ \& \ x_j$$
$$\text{respectively}.$$

Then $\varphi(\hat{\gamma}_i)$ & $\varphi(\hat{\gamma}_j)$ are geodesics in $B_\delta^N$ joining $\varphi(\zeta)$ and $\varphi(x_i) = y = \varphi(x_j)$.

$\Rightarrow \varphi(\widehat{\gamma_i}) = \varphi(\widehat{\gamma_j}) = \gamma$ the unique geodesic in $B_\delta^N$ joining $\varphi(\zeta)$ to $y$.

Therefore $\widehat{\gamma_i}, \widehat{\gamma_j}$ are both liftings of $\gamma$ passing thro. a common point $Z$, we have $\widehat{\gamma_i} = \widehat{\gamma_j}$.

$\Rightarrow x_i = \widehat{\gamma_i}(1) = \widehat{\gamma_j}(1) = x_j$ <u>Contradiction</u> !

This completes the proof of (iii) ✕✕

By the claim, $B_\delta^N$ is the required uniform nbd of $y$.

∴ $\varphi$ is covering. ✕✕

Lemma 9: Let
- $M =$ complete Riemannian mfd.
- $x \in M$ s.t,
- $\exp_x : T_x M \to M$ has no conjugate point.

Then $\exp_x$ is a covering map.

Pf: Let $g =$ Riem. metric of $M$.

Denote $\tilde{g} = (\exp_x)^* g$ be the pull-back metric of $g$ by $\exp_x$ on $T_x M$.

Claim: $\tilde{g}$ is a complete metric on $T_x M$.

Pf of Claim: Note that Euclidean rays (from 0)

in $T_xM$ can be parametrized by

$$\tilde{\gamma} : [0, \infty) \longrightarrow T_xM$$
$$t \longmapsto tv$$

for some $v \in T_xM$

By definition of $\exp_x$, $\exp_x(\tilde{\gamma}(t))$ is a geodesic in $M$ starting at $x$. Therefore, by definition of $\tilde{g} = (\exp_x)^* g$, $\tilde{\gamma}(t)$ is a geodesic of $\tilde{g}$ starting from $0$. This implies geodesics from $0 \in T_xM$ is defined $\forall t \in [0, \infty)$. Hence

$$\exp_0 : T_0(T_xM) \rightarrow (T_xM, \tilde{g}) \text{ is defined on}$$

the whole $T_0(T_xM)$. Therefore, Hopf-Rinow Thm

$\Rightarrow (T_xM, \tilde{g})$ is complete. This completes the

proof of the claim.

Now by the claim and the assumption that

$\exp_x$ has no conjugate point,

$$\exp_x : (T_xM, \tilde{g}) \longrightarrow (M, g)$$

is a local isometry from a complete Riem. mfd.

$\therefore$ Lemma 8 $\Rightarrow \exp_x : T_xM \rightarrow M$ is a covering

�

# Pfof (2) of Cartan-Hadamard :

By Lemma 9, $\exp_x : T_x M \to M$ is a covering.
Together with the assumption that $M$ is
simply-connected, we have proved that
$\exp_x$ is a diffeomaphism. $\#$

Thm 10 : Let $M, N =$ Simply-connected $n$-dim'l space form
with constant sectional curvature $k$.
Let $x \in M$ & $y \in N$ and

$$\{e_1, \cdots, e_n\} \subset T_x M \quad \& \quad \{\varepsilon_1, \cdots, \varepsilon_n\} \subset T_y N$$

are orthonormal basis respectively.

Then $\exists$ unique isometry $\varphi: M \to N$ such that

$$\begin{cases} \varphi(x) = y \quad \text{and} \\ d\varphi(e_i) = \varepsilon_i \quad \forall i \end{cases}.$$

Note: Thm 10 $\Rightarrow$ uniqueness of the Thm 1 of Ch5.

To prove this Thm, we need the following Lemmas:

Lemma 11: Let
- $M = n$-dim'l space form
- Constant sectional curvature $K$

- $x \in M$, $\{e_1, \ldots, e_n\} \subset T_x M$ ortho. basis.

Then the curvature tensor satisfies

$$R_{e_i e_j} e_k = K(\delta_{ik} e_j - \delta_{jk} e_i), \quad \forall i,j,k = 1, \ldots, n.$$

Pf: Define $\tilde{R}$ by the L.H.S. ie

$$\tilde{R}_{e_i e_j} e_k \overset{\text{def}}{=} K(\delta_{ik} e_j - \delta_{jk} e_i)$$

Then $\tilde{R}$ can be extended to a tensor (Ex) satisfying all the symmetric properties of the curvature tensor (i.e. (1)-(4) in Lemma 1 of §3.3)

(Ex).     Furthermore, for tangent vectors $v$ & $w$

with    $|v| = |w| = 1$ & $\langle v, w \rangle = 0$,

one has    $\langle \hat{R}_{vw} v, w \rangle = K$. (Ex)

Therefore (Lemma 2 of §3.3), we have

$$\hat{R} \equiv R. \qquad \text{※}$$

Lemma 12 : Same assumption as in Lemma 11.

Let • $v \in T_x M$ with $|v| = 1$

• $v^{\perp} =$ orthogonal complement of $v$.

Then
$$R_{U} w U = \begin{cases} Kw, & \text{if } w \in U^{\perp} \\ 0, & \text{if } w = cv \text{ for some } c \in \mathbb{R} \end{cases}$$

(Pf: Straight faward from Lemma 11)