<u>Prop</u>: Let $K \geq +1$, then $c(r) \leq 2\pi \sin r$, for small r.

<u>Pf</u>: Consider a comparison function $h(t) = \sin t$

Then $\begin{cases} h'' + h = 0 \\ h(0) = 0 \\ h'(0) = 1. \end{cases}$

$$\Rightarrow \quad (hf' - fh')' = hf'' - fh''$$
$$= -Kfh + fh$$
$$= -(K-1)fh.$$

Since $f(0) = h(0) = 0$, $f'(0) = h'(0) = 1$, we have

$f \geq 0$, $h \geq 0$ for small $t > 0$.

$$\Rightarrow \quad (hf' - fh')' \le 0 \quad \text{for small } t > 0$$

$$\Rightarrow \quad hf' - fh' \le h(0)f'(0) - f(0)h'(0) = 0 \quad \text{for small } t > 0$$

$$\Rightarrow \quad \left(\frac{f}{h}\right)' = \frac{hf' - fh'}{h^2} \le 0 \quad \text{for small } t > 0$$

$$\Rightarrow \quad \frac{f(t)}{h(t)} \le \lim_{\varepsilon \to 0} \frac{f(\varepsilon)}{h(\varepsilon)} = \frac{f'(0)}{h'(0)} = 1$$

$$\Rightarrow \quad f(t) \le h(t) = \sin t \quad \text{for small } t > 0$$

Hence $\quad C(r) = \int_0^{2\pi} f(r)\, d\theta \le 2\pi \sin r$. #

for small $r > 0$

Prop: If $K \le -1$, we have $C(r) \ge 2\pi \sinh(r)$.

(for small r at this moment)

Pf: Consider $h(t) = \sinh t$.

Then $\begin{cases} h''(t) - h(t) = 0 \\ h(0) = 0 \\ h'(0) = 1 \end{cases}$

$$\Rightarrow \quad (hf' - fh')' = hf'' - fh''$$

$$= -Kfh - fh$$

$$= -(K+1) fh$$

$$\ge 0 \quad \text{for small } t > 0$$

$$\Rightarrow \quad h f' - f h' \geq h(0) f'(0) - f(0) h'(0) = 0$$

$$\Rightarrow \quad \left(\frac{f}{h}\right)' \geq 0$$

$$\Rightarrow \quad \frac{f(t)}{h(t)} \geq \lim_{\varepsilon \to 0} \frac{f(\varepsilon)}{h(\varepsilon)} = \frac{f'(0)}{h'(0)} = 1$$

$$\Rightarrow \quad f(t) \geq \sinh(t) \quad \text{for small } t > 0$$

$$\Rightarrow \quad C(r) \geq 2\pi \sinh(r) \quad \text{for small } r > 0.$$

#

Ch6 Jacobi field, Cartan-Hadamard Thm

§6.1 Jacobi field

Let γ = normalized geodesic (i.e. $|\gamma'|=1$)

Recall that the Jacobi eqt (for vector fields along γ)

is

$$U'' + R_{\gamma' U}\gamma' = 0 \qquad \text{(Jac)}$$

where $U'' = D_{\gamma'} D_{\gamma'} U \quad (U' = D_{\gamma'} U)$

Let $\{e_1(t), \cdots, e_n(t)\}$ be parallel vector fields along γ

s.t. $\forall t$

$$\begin{cases} e_1(t) = \gamma'(t) \\ \{e_i(t)\}_{i=1}^n \text{ is an athonormal basis of } T_{\gamma(t)}M. \end{cases}$$

Then $\forall$ vector field U along γ, we write

$$U(t) = \sum_{i=1}^n f^i(t) e_i(t), \quad \text{for some functions } f^i(t).$$

Similarly, the curvature can be written as

$$R_{e_i(t)e_j(t)} e_k(t) = \sum_{l=1}^n R_{ijk}^l(t)\, e_l(t),$$

where $R_{ijk}^l(t) = \langle R_{e_i(t)e_j(t)} e_k(t), e_l(t) \rangle$

Then the eqt (Jac) $\Rightarrow$

$$0 = V'' + R_{\gamma' V}\gamma'$$

$$= \left(\sum_i f^i e_i\right)'' + R_{e_1 V} e_1$$

$$= \sum_i (f^i)'' e_i + R_{e_1 (\sum_l f^l e_l)} e_1$$

$$= \sum_i (f^i)'' e_i + \sum_l f^l R_{e_1 e_l} e_1$$

$$= \sum_i (f^i)'' e_i + \sum_l f^l \sum_i R^i_{1l1} e_i$$

$$= \sum_i \left[(f^i)'' + \sum_l R^i_{1l1} f^l\right] e_i$$

$\therefore$ (Jac) $\Leftrightarrow$ $\boxed{(f^i)'' + \sum_l R^i_{1l1} f^l = 0 \quad \forall\, i = 1, \cdots, n}$

which is a 2^{nd} order linear ODE system.

Lemma

(1) Let γ be a geodesic. Then given any $v, w \in T_{\gamma(0)}M$,

$\exists$ a unique Jacobi field $U(t)$ along γ s.t.

$$\begin{cases} U(0) = v \\ U'(0) = w. \end{cases}$$

(2) Unless $U \equiv 0$, the zero set of $U(t)$ along γ is discrete.

(Pf: ODE theory)

Lemma 2 : Let U be a vector field along a normalized geodesic γ. Then

U is a Jacobi field along γ

$\Leftrightarrow$ U is the transversal vector field of a one-parameter family of geodesics.

Pf: ($\Leftarrow$) Proved in previous chapter.

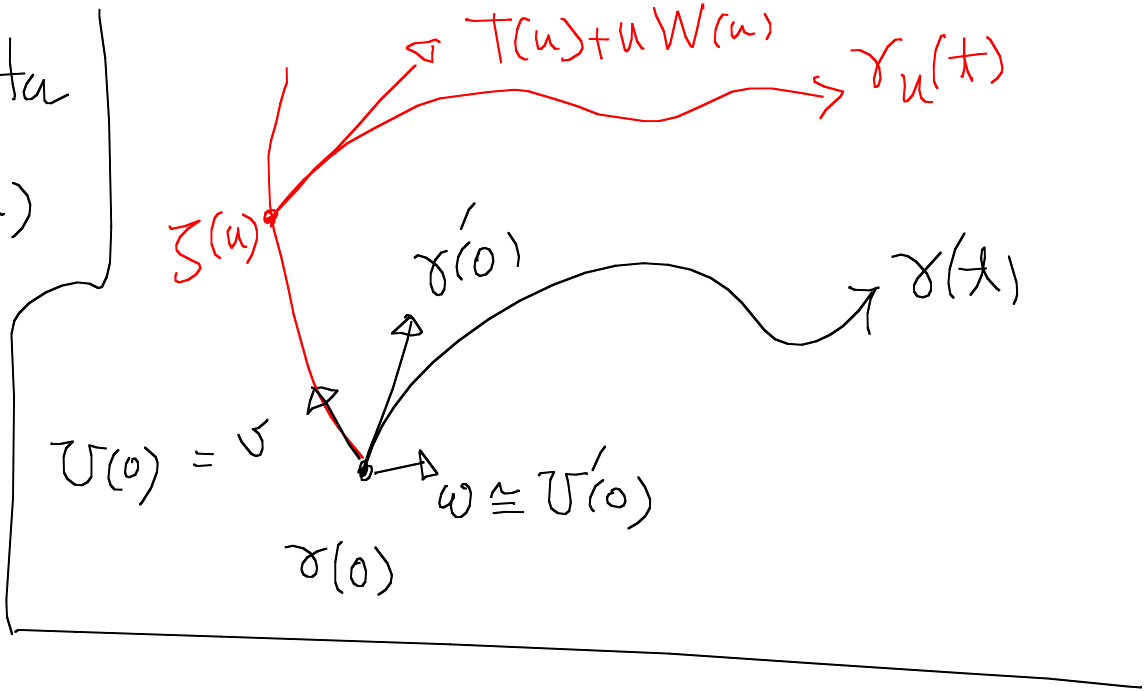
($\Rightarrow$) Let $v = U(0)$ & $w = U'(0)$

(by identifying $T_{\vec{p}}(T_{\gamma(0)}M) \cong T_{\gamma(0)}M$, $\forall \vec{p}$)

And let $\zeta : [0, \varepsilon] \to M$ be a geodesic

s.t. $\zeta(0) = \gamma(0)$ & $\zeta'(0) = v$

Define parallel vector fields $T(u)$ & $W(u)$ for $u \in [0, \varepsilon]$, along ζ such that



$$T(0) = \gamma'(0) \quad \& \quad W(0) = \omega$$

$\forall\, u \in [0, \varepsilon]$, define

$$\Gamma(t,u) = \gamma_u(t) = \exp_{\zeta(u)}\left[t\left(T(u) + uW(u)\right)\right].$$

Let V_1 = transversal vector field of γ_u along $\gamma = \gamma_0$.

Then V_1 is a Jacobi field.

Since $U_1(0) = \frac{\partial}{\partial u}\Big|_{u=0} \gamma_u(0)$

$$= \frac{\partial}{\partial u}\Big|_{u=0} \exp_{\zeta(u)}(0)$$

$$= \frac{\partial}{\partial u}\Big|_{u=0} \zeta(u) = \zeta'(0) = v.$$

Since $T_1 = d\Gamma\left(\frac{\partial}{\partial t}\right)$ is a vector field along Γ & when restricted to γ, we have

$$[T_1, U_1] = 0.$$

And hence

$$U_1'(0) = D_{\gamma'(0)} U_1 = D_{U_1(0)} T_1 \quad \left(\text{since } [T_1, U_1] = 0\right)$$

$$= D_v T_1 = D_{\zeta'(0)} T_1$$

Note that $T_1(\zeta(u)) = \left.\dfrac{\partial}{\partial t}\right|_{t=0} \exp_{\zeta(u)}\left[t\left(T(u)+uW(u)\right)\right]$

$$= T(u) + u W(u)$$

$\therefore \quad U_1'(0) = D_{\zeta'(0)} T_1 = D_{\zeta'(0)}\left[T(u)+uW(u)\right]$

$= W(0)$ (since T, W are parallel along ζ)

$= w$

Altogether $U(0) = U_1(0)$ & $U'(0) = U_1'(0)$,

Uniqueness of Jacobi field $\Rightarrow U = U_1$

= transversal vector field $\boxtimes$

Lemma 3: Let U be a Jacobi field along a geodesic γ.

Then $\exists\, a, b \in \mathbb{R}$ such that

$$U = U^{\perp} + (at+b)\gamma',$$

where $U^{\perp}$ is a Jacobi field s.t. $\langle U^{\perp}, \gamma' \rangle = 0 \ \forall t$.

Pf: Consider

$$\frac{d^2}{dt^2}\langle U, \gamma' \rangle = \frac{d}{dt}\left(D_{\gamma'} \langle U, \gamma' \rangle \right)$$

$$= \frac{d}{dt}\left(\langle U', \gamma' \rangle + \langle U, \cancel{D_{\gamma'}\gamma'} \rangle \right)$$

$$= \langle U'', \gamma' \rangle + 0$$

$$= -\langle R_{\gamma' V}\gamma', \gamma'\rangle = 0$$

$\Rightarrow \langle V, \gamma'\rangle = \tilde{a}\,t + \tilde{b}$ for some $\tilde{a}, \tilde{b} \in \mathbb{R}$.

Let $V^{\perp} = V - \langle V, \frac{\gamma'}{|\gamma'|}\rangle \frac{\gamma'}{|\gamma'|}$

$$= V - \left(\frac{\tilde{a}}{|\gamma'|^2}t + \frac{\tilde{b}}{|\gamma'|^2}\right)\gamma'$$

Since $|\gamma'| \equiv$ const., $V^{\perp} = V - (at+b)\gamma'$

with $a = \frac{\tilde{a}}{|\gamma'|^2}$, $b = \frac{\tilde{b}}{|\gamma'|^2} \in \mathbb{R}$.

and satisfies $\langle V^{\perp}, \gamma'\rangle = 0$.

$$(V^\perp)'' = V'' - [(at+b)\gamma']''$$

$$= V'' = -R_{\gamma' V}\gamma'$$

$$= -R_{\gamma' V^\perp}\gamma' - (at+b)R_{\cancel{\gamma}\gamma'}\gamma'$$

$$= -R_{\gamma' V^\perp}\gamma'$$

$\Rightarrow V^\perp$ is a Jacobi field. #

<u>Lemma 4</u> If V is a Jacobi field along a geodesic γ such that

$$\langle V(t_1), \gamma'(t_1)\rangle = \langle V(t_2), \gamma'(t_2)\rangle = 0$$

for 2 different t_1 & t_2. Then $\langle V(t), \gamma'(t)\rangle = 0$, $\forall t$.

(Pf: Since $\langle J(t), \gamma'(t)\rangle$ is linear in t)

In summary, we have the following facts of Jacobi fields:

(A) Let $\zeta: [0, \varepsilon] \to M$ be a curve in M,

$u \mapsto \zeta(u)$

$T(u)$, $W(u)$ parallel vector fields along ζ.

Then

$$\gamma_u(t) = \exp_{\zeta(u)}\left[t\left(T(u) + uW(u)\right)\right]$$

determines a 1-para. family of geodesics $\{\gamma_u\}$ s.t. its transversal vector field $J(t)$ along γ_0

is a Jacobi field with $\begin{cases} U(0) = \zeta'(0) \\ U'(0) = W(0) \end{cases}$.

(B) (If we take $\zeta(u) \equiv x \in M$ (constant curve) in (A), then we have)

$\forall x \in M$; T, $\omega \in T_xM$. Then the 1-para. family of geodesics $\{\gamma_u\}$ defined by

$$\gamma_u(t) = \exp_x\left[t\left(T + u\,\omega\right)\right]$$

has a transversal vector field $U(t)$ s.t. $U(t)$ is a Jacobi field with

$$\begin{cases} U(0) = 0 \\ U'(0) = w. \end{cases}$$

(C′) [Furthermore, adding condition $\langle T, w \rangle = 0$ to (B)]

Let $x \in M$; $T, w \in T_x M$ s.t. $\langle T, w \rangle = 0$.

Let $\gamma_u(t) = \exp_x [t(T + u w)]$,

Then the transversal vector field $U(t)$ of $\{\gamma_u\}$

is a <u>normal</u> Jacobi field with

$$\begin{cases} U(0) = 0 \\ U'(0) = w. \end{cases}$$

(normal Jacobi field := Jacobi field normal to the geodesic)

The proof of (C) needs (extension of) Gauss Lemma.

Lemma 5 (Gauss lemma)

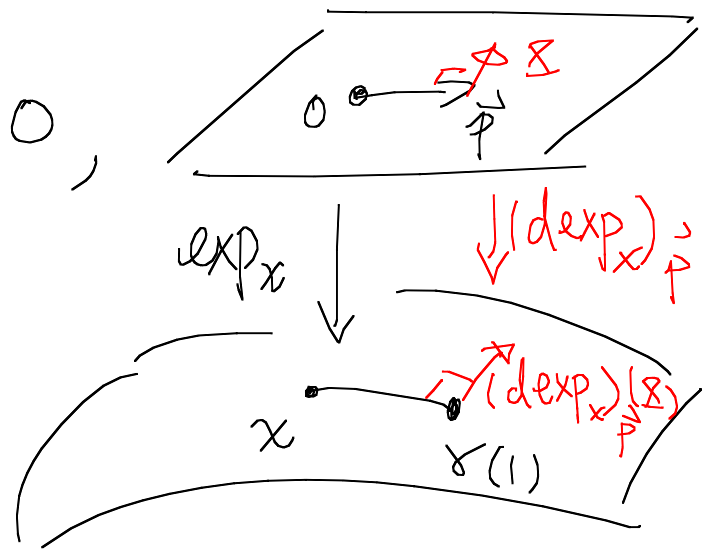
Let M complete, $x \in M$, $\vec{p} \in T_xM$, $\xi \in T_{\vec{p}}(T_xM) \cong T_xM$

If $\langle \vec{p}, \xi \rangle = 0$, then

$$\langle (d\exp_x)_{\vec{p}}(\xi), \gamma'(1) \rangle = 0,$$

where $\gamma : [0,1] \to M$

$$t \mapsto \exp_x(t\vec{p}).$$



Pf: Let $\xi : [0, \varepsilon] \to T_x M$ be a curve in $T_x M$ s.t.

$\xi(0) = \vec{p}$, $\xi'(0) = \underline{X}$; and that

$$\xi([0, \varepsilon]) \subset S^{n-1}_{|\vec{p}|} \subset T_x M.$$

Such ξ exists since $\underline{X} \perp \vec{p}$ (ie $\underline{X} \in T_{\vec{p}} S^{n-1}_{|\vec{p}|}$)

Consider $\Gamma : [0, 1] \times [0, \varepsilon] \longrightarrow M$

$$\begin{array}{c} \cup \\ (t, u) \end{array} \mapsto \exp_x [t\,\xi(u)]$$

Let $T = d\Gamma\left(\frac{\partial}{\partial t}\right)$ & $U = d\Gamma\left(\frac{\partial}{\partial u}\right)$.

Then $\gamma(t) = \Gamma(t, 0)$,

$\gamma'(1) = T(\gamma(1))$

$$\left(d\exp_x\right)_{\vec{p}}(X) = U(\gamma(1)).$$

Since $|\xi(u)| = |\vec{p}|$

$\Rightarrow \quad \langle T, T\rangle = |\vec{p}|^2$ (geodesic has const. speed)

$\therefore \quad T\langle U, T\rangle = \langle D_T U, T\rangle + \langle U, \cancel{D_T T}\rangle$ (γ = geodesic)

$= \langle D_U T + \cancel{[T, U]}, T\rangle$ $\left([T,U] = d\Gamma\left[\frac{\partial}{\partial t}, \frac{\partial}{\partial u}\right]\right)$

$= \langle D_U T, T\rangle = \frac{1}{2} U\langle T, T\rangle$

$= 0.$

$\Rightarrow \quad \langle U, T\rangle$ = constant along γ

$= \lim_{t\to 0} \langle U(t), T(t)\rangle = \langle \cancel{U}(0), T(0)\rangle = 0$ #

Pf of (C) : Let $\zeta : [0, \varepsilon] \to T_x M$

$u \mapsto t(T + u\omega)$

assumption

Then $\langle \zeta'(0), \zeta(0) \rangle = \langle t\omega, tT \rangle = t^2 \langle \omega, T \rangle = 0$,

and $(d\exp_x)_{(tT)}(\zeta'(0)) = U(t)$. ($U$ = transversal vector field of $\exp_x(t(T+u\omega))$)

Consider the curve $\gamma : [0,1] \to M$

$\tau \mapsto \exp_x(\tau(tT))$.

(Then $\gamma_0(t) = \exp_x(tT)$)

$$\Rightarrow \gamma'(1) = \frac{d}{d\tau}\Big|_{\tau=1} [\exp_x(\tau t T)] = (d\exp_x)_{(tT)}(tT)$$

$$= t\,(d\exp_x)_{(tT)}(T)$$

$$= t\,\gamma_0'(t) \qquad (\text{" ' " means derivative wrt } t)$$

Applying the Gauss Lemma to γ and $X = \zeta'(0)$,

$$\vec{p} = \zeta(0) = tT \qquad (\langle \vec{X}, \vec{p} \rangle = \langle \zeta'(0), \zeta(0) \rangle = 0),$$

we have

$$\langle U(t), \gamma_0'(t) \rangle = \langle (d\exp_x)_{(tT)}(\zeta'(0)),\ \frac{1}{t}\gamma'(1) \rangle$$

$$= \frac{1}{t} \langle (d\exp_x)_{(tT)}(X),\ \gamma'(1) \rangle = 0$$

$\Rightarrow U$ is normal. Other conclusions are clear from (B) ☒