3.2 Curvature Tensor

Let $\mathcal{T}^* =$ Algebra of tensor fields on M / $C^\infty(M)$

Then $\forall$ vector field $X \in \Gamma(M)$,

$$D_X : \mathcal{T}^* \longrightarrow \mathcal{T}^* \text{ is a derivation.}$$

Therefore, if we have D_X & D_Y, the Lie bracket

$$[D_X, D_Y] = D_X D_Y - D_Y D_X$$

is also a derivation (Ex.)

Hence we can make the following definition

$$R_{XY} = D_{[X,Y]} - [D_X, D_Y]$$

$$= -D_X D_Y + D_Y D_X + D_{[X,Y]}$$

<u>Prop</u>:

(1) $R_{XY} : \mathfrak{T}^* \to \mathfrak{T}^*$ is a derivation

(2) R_{XY} preserves the type of a tensor field,

i.e. K is (r,s)-type $\Rightarrow$ $R_{XY}K$ is also (r,s)-type.

(3) $\forall f \in C^\infty(M)$

$$R_{(fX)Y} K = R_{X(fY)} K = R_{XY}(fK) = f R_{XY} K$$

(4) $\forall f \in C^\infty(M)$, $R_{XY} f = 0$

Pf: We check only $R_{(fX)Y}K = fR_{XY}K$.

$$R_{(fX)Y}K = -D_{fX}D_YK + D_YD_{fX}K + D_{[fX,Y]}K$$

$$= -fD_XD_YK + D_Y(fD_XK) + D_{[fX,Y]}K$$

$$= -fD_XD_YK + fD_YD_XK + (Yf)D_XK + D_{[fX,Y]}K$$

$$= fR_{XY}K - fD_{[X,Y]}K + (Yf)D_XK + D_{[fX,Y]}K$$

Note that $[fX, Y] = fXY - Y(fX)$

$$= f(XY - YX) - (Yf)X = f[X,Y] - (Yf)X$$

$\Rightarrow \quad R_{(fX)Y}K = fR_{XY}K$. #

($\therefore$ $D_{[X,Y]}$ is needed in the definition in order to have property (3))

Note: By property (3), if $K = Z$ is also a vector field then one can use $R_{XY}Z$ to define a (1,3)-tensor:

$$(\omega, X, Y, Z) \xmapsto{R} \omega(R_{XY}Z) \qquad \forall \text{ 1-form } \omega, \ \& \ X, Y, Z \in \Gamma(M)$$

It also defines a (0,4)-tensor R (using metric g)

$$R(X,Y,Z,W) = g(R_{XY}Z, W), \quad \forall X, Y, Z, W \in \Gamma(M).$$

Def: $R_{XY}Z$ & $R(X,Y,Z,W)$ are called the (Riemannian) curvature tensor of g (More precisely, R is the curvature tensor of g.)

Local famula: In a coordinate system $(x^1, \cdots, x^n)$

if $g_{ij} = g\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right)$ and

$$\Gamma_{ij}^{k} = \frac{1}{2} g^{kl}\left(\frac{\partial g_{lj}}{\partial x^i} + \frac{\partial g_{il}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^l}\right) \quad \text{(Christoffel symbol)}$$

then $R_{ijkl} \overset{\text{def}}{=} R\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^k}, \frac{\partial}{\partial x^l}\right)$ is given by

$$\boxed{\begin{aligned} R_{ijkl} = \frac{1}{2}\left(\frac{\partial^2 g_{il}}{\partial x^j \partial x^k} + \frac{\partial^2 g_{jk}}{\partial x^i \partial x^l} - \frac{\partial^2 g_{ik}}{\partial x^j \partial x^l} - \frac{\partial^2 g_{jl}}{\partial x^i \partial x^k}\right) \\ + \left(g_{rs}\Gamma_{jk}^{r}\Gamma_{il}^{s} + g_{rs}\Gamma_{jl}^{r}\Gamma_{ik}^{s}\right) \end{aligned}}$$

(Pf: Omitted) $\therefore$ R is a 2^{nd} order non-linear function of g.

__Def__ : Let (M, g) & (N, h) be 2 Riemannian manifolds.

A C^∞ map $\varphi : M \to N$ is called a __local__ __isometry__

$\iff \forall x \in M$

$$d\varphi : (T_xM, g_x) \to (T_{\varphi(x)}N, h_{\varphi(x)})$$

is an isometry of the inner product spaces.

i.e. $\forall v_1, v_2 \in T_xM$,

$$h_{\varphi(x)}(d\varphi(v_1), d\varphi(v_2)) = g_x(v_1, v_2)$$

Note : If φ = local isom, then $\dim M = \dim N$,

and φ is an immersion.

__Def__ : $\varphi : (M, g) \to (N, h)$ is called a __global__ __isometry__, or simply an <u>isometry</u>,

$\Leftrightarrow$ φ is a local isometry & a diffeomorphism.

<u>Fact</u>: Let

- $\varphi : (M, g) \to (M', g')$ is an isometry
- D = Levi-Civita connection of g
- D' = " " " g'
- $X, Y \in \Gamma(M)$ & $X', Y' \in \Gamma(M')$

  s.t. $d\varphi(X) = X'$, $d\varphi(Y) = Y'$

Then $d\varphi(D_X Y) = D'_{X'} Y'$

$\therefore$ Levi-Civita connection is a <u>metric</u> <u>invariant</u>.

(Pf: Ex)

Thm (Metric invariance of curvature tensor)

Let
- $\varphi : (M, g) \to (M', g')$ is an isometry.
- R, R' = curvature tensors of g & g' respectively
- $X, Y, Z, W \in \Gamma(M)$, $X', Y', Z', W' \in \Gamma(M')$ s.t.
  $d\varphi(X) = X'$, $d\varphi(Y) = Y'$, $d\varphi(Z) = Z'$, $d\varphi(W) = W'$.

Then
- $d\varphi(R_{XY} Z) = R'_{X'Y'} Z'$
- $R(X, Y, Z, W) = R'(X', Y', Z', W') \circ \varphi$

(Pf : Ex.)

Note: If $\dim M = 2$, then one can define the Gaussian curvature $K : M \to \mathbb{R}$ by

$$K(x) = R(e_1, e_2, e_1, e_2)(x), \quad \forall x \in M$$

for any orthonormal basis $\{e_1, e_2\}$ of $T_x M$.

And this K coincides with original definition for $M^2 \subset \mathbb{R}^3$.

Def: A Riemannian manifold (M, g) is called flat if its curvature tensor $R \equiv 0$.

eg $(\mathbb{R}^n, \text{standard metric}) = (\mathbb{R}^n, dx^1 \otimes dx^1 + \cdots + dx^n \otimes dx^n)$

is flat. (Reason: $g_{ij} \equiv$ const. $\Rightarrow \Gamma_{ij}^k = 0 \Rightarrow R \equiv 0$)

3.3 Basic properties of curvature tensor

Lemma1: $\forall$ vector fields X, Y, Z, W

(1) $R_{XY} = -R_{YX}$

(2) (1st Bianchi identity)

$$R_{XY}Z + R_{YZ}X + R_{ZX}Y = 0$$

(3) $R(X, Y, Z, W) = -R(X, Y, W, Z)$

(4) $R(X, Y, Z, W) = R(Z, W, X, Y)$

Pf: (1) is clear.

To prove (2) & (3), we only need to check the case that $\{X, Y, Z, W\} = \left\{\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^k}, \frac{\partial}{\partial x^l}\right\}$

(since R is a tensor)

In this case $[X, Y] = 0$, ...

Hence
$$\begin{cases} D_X Y = D_Y X \\ R_{XY} = -D_X D_Y + D_Y D_X \end{cases}$$

$$\Rightarrow \quad R_{XY} Z + R_{YZ} X + R_{ZX} Y$$

$$= (-D_X D_Y Z + D_Y D_X Z) + (-D_Y D_Z X + D_Z D_Y X)$$
$$+ (-D_Z D_X Y + D_X D_Z Y)$$

$$= D_X(-D_Y Z + D_Z Y) + D_Y(D_X Z - D_Z X)$$
$$+ D_Z(D_Y X - D_X Y)$$

$$= 0.$$

This proves (2).

For (3), we first note that

$$R(X,Y,Z,Z) = \langle R_{XY} Z, Z \rangle$$
$$= \langle -D_X D_Y Z + D_Y D_X Z, Z \rangle$$
$$= -X\langle D_Y Z, Z\rangle + \langle D_Y Z, D_X Z\rangle$$
$$+ Y\langle D_X Z, Z\rangle - \langle D_X Z, D_Y Z\rangle$$
$$= -X\left(Y\left(\tfrac{1}{2}\langle Z,Z\rangle\right)\right) + Y\left(X\left(\tfrac{1}{2}\langle Z,Z\rangle\right)\right)$$

$$= -\frac{1}{2}[X,\cancel{Y}](\langle Z,Z\rangle) = 0$$

Hence $\forall \{X, Y, Z, W\}$ with $[X,Y]=0$, ...

we have $0 = R(X, Y, Z+W, Z+W)$

$$= R(X,\cancel{Y},Z,Z) + R(X,Y,Z,W) + R(X,Y,W,Z)$$
$$+ R(X,\cancel{Y},W,W)$$

$$\Rightarrow \quad R(X,Y,Z,W) = -R(X,Y,W,Z).$$

This proves (3).

<u>Proof of (4)</u> (Jost)

$$R(X,Y,Z,W) = -R(Y,X,Z,W) \qquad \text{by (1)}$$

1st Bianchi

$$= R(Z,Y,X,W) + R(X,Z,Y,W)$$

Similarly

$$R(X,Y,Z,W) = -R(X,Y,W,Z) \quad \text{by (3)}$$

$$= R(Y,W,X,Z) + R(W,X,Y,Z)$$

$\Rightarrow$

$$2R(X,Y,Z,W) = R(Z,Y,X,W) + R(X,Z,Y,W) \quad \text{--- } (*)$$

$$+ R(Y,W,X,Z) + R(W,X,Y,Z)$$

Similarly

$$2R(Z,W,X,Y) = R(X,W,Z,Y) + R(Z,X,W,Y)$$

$$+ R(W,Y,Z,X) + R(Y,Z,W,X)$$

by (1) & (3) $$= + R(W,X,Y,Z) + R(X,Z,Y,W)$$

$$+ R(Y,W,X,Z) + R(Z,Y,X,W)$$

by (*) $= 2R(X, Y, Z, W)$

$\#$

<u>Lemma 2</u> Let $Q(X,Y) \stackrel{\text{def}}{=\!=} R(X,Y,X,Y)$.

Then Q determines R.

i.e. if R, R' are tensor fields satisfying (1) — (4) in lemma 1, then

$$Q = Q' \Rightarrow R = R'$$

(Pf: Omitted)

<u>Def</u>: Let
- π be a 2-dim'l subspace in T_xM
- $\{v_1, v_2\}$ = basis of π

then

$$K(\Pi) = \frac{R(v_1, v_2, v_1, v_2)}{|v_1 \wedge v_2|^2}$$

where $|v_1 \wedge v_2|^2 = \det(\langle v_i, v_j \rangle)$ $\quad i,j = 1,2$

$$= |v_1|^2 |v_2|^2 - \langle v_1, v_2 \rangle^2 .$$

is called the <u>sectional curvature of Π</u>.

<u>Note</u>:
- $K(\Pi)$ doesn't depend on the basis $\{v_1, v_2\}$
- If $\{e_1, e_2\}$ = orthonormal basis of Π, then

$$K(\Pi) = R(e_1, e_2, e_1, e_2)$$

- Lemma 2 $\Rightarrow$ K determines R
- Sectional curvature K is a metric invariant

i.e. If $\varphi: M \to M'$ = isometry,

$\Pi \subset T_xM$, $\Pi' \subset T_{\varphi(x)}M'$ are 2-dim'l

subspaces with

$$d\varphi(\Pi) = \Pi'$$

Then $K(\Pi) = K'(\Pi')$.

e.g. If $K(\Pi) \equiv 0$ $\forall x$ & $\Pi^2 \subset T_xM$, then $R \equiv 0$

Lemma 3 (The 2nd Bianchi Identity)

$$\boxed{(D_X R)_{YZ} + (D_Y R)_{ZX} + (D_Z R)_{XY} = 0}, \quad \forall X, Y, Z \in \Gamma(TM)$$

Pf: It is sufficient to prove the identity for vector fields satisfying $[X, Y] = \cdots = 0$.

For these vector fields $\begin{cases} D_X Y = D_Y X \\ R_{XY} = -D_X D_Y + D_Y D_X \end{cases}$

By definition

$$(D_X R)_{YZ} W = D_X(R_{YZ} W) - R_{(D_X Y) Z} W - R_{Y (D_X Z)} W$$
$$- R_{YZ}(D_X W)$$

$$(D_Y R)_{ZX} W = D_Y(R_{ZX} W) - R_{(D_Y Z) X} W - R_{Z (D_Y X)} W$$
$$- R_{ZX}(D_Y W)$$

$$(D_Z R)_{XY} W = D_Z (R_{XY} W) - R_{(D_Z X)Y} W - R_{X(D_Z Y)} W$$

$$- R_{XY}(D_Z W)$$

$$\Rightarrow (D_X R)_{YZ} W + (D_Y R)_{ZX} W + (D_Z R)_{XY} W$$

$$= D_X\left(-\cancel{D_Y D_Z W}^{\,1} + \cancel{D_Z D_Y W}^{\,2}\right) + D_Y\left(-D_Z D_X W + D_X D_Z W\right)$$

$$+ D_Z\left(-D_X D_Y W + D_Y D_X W\right)$$

$$-\left(-D_Y D_Z + D_Z D_Y\right)(D_X W) - \left(-D_Z D_X + \cancel{D_X D_Z}^{\,2}\right)(D_Y W)$$

$$-\left(-\cancel{D_X D_Y}^{\,1} + D_Y D_X\right)(D_Z W)$$

$$- \cancel{R_{(D_X Y)Z}}^{\,a} W - \cancel{R_{Y(D_X Z)}}^{\,b} W - \cancel{R_{(D_Y Z)X}}^{\,c} W - \cancel{R_{Z(D_Y X)}}^{\,a} W$$

$$- R_{\cancel{(D_Z X)}^{\,b} Y} W - R_{X \cancel{(D_Z Y)}^{\,c}} W \qquad \left(\text{using } \begin{array}{l} D_X Y = D_Y X \cdots \\ R_{XY} = -R_{YX} \cdots \end{array} \right)$$

$$= 0$$

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Lemma 4 (Ricci Identity)

$$\boxed{D^2 T(\dots, X, Y) - D^2 T(\dots, Y, X) = (R_{XY} T)(\dots)}$$

$\forall$ tensor field T

(Therefore, $R_{XY} = D^2_{XY} - D^2_{YX}$)

Pf: $D^2 T(\dots, X, Y)$

$$= (D_Y(DT))(\dots, X)$$

$$= D_Y\left[(DT)(\cdots, X)\right] - \sum (DT)(\cdot\, D_Y \cdot \cdot, X) - (DT)(\cdots, D_Y X)$$

$$= D_Y\left[(D_X T)(\cdots)\right] - \sum (D_X T)(\cdot\, D_Y \cdot \cdot) - (D_{D_Y X} T)(\cdots)$$

$$= (D_Y D_X T)(\cdots) - (D_{D_Y X} T)(\cdots)$$

$$= (D_Y D_X T - D_{D_Y X} T)(\cdots)$$

Hence $(D^2 T)(\cdots, X, Y) - (D^2 T)(\cdots, Y, X)$

$$= \left[(D_Y D_X T - D_{D_Y X} T) - (D_X D_Y T - D_{D_X Y} T)\right](\cdots)$$

$$= \left[(-D_X D_Y + D_Y D_X + D_{(D_X Y - D_Y X)})T\right](\cdots)$$

$$= \left[(-D_X D_Y + D_Y D_X + D_{[X,Y]})T\right](\cdots)$$

$= (R_{XY} T)(\cdots)$ ##

3.4 Various notions of curvature

Def : The Ricci tensor "Ric" is the (0,2)-tensor field defined by

$$Ric(X,Y) = \sum_{i=1}^{n} R(e_i, X, e_i, Y) \quad , \forall X, Y \in \Gamma(TM)$$

where $\{e_i\}$ = orthonormal basis of $T_x M$.

Note :
- Ric does not depend on the o.n. basis $\{e_i\}$
- Ric is symmetric, i.e. $Ric(X,Y) = Ric(Y,X)$.

Def: Let $X \in T_xM$ with $|X| = 1$. Then $Ric(X, X)$ is called the Ricci curvature in the direction of X.

Note: One can choose an o.n basis $\{e_1, \dots, e_n\}$ of T_xM such that $e_1 = X$. Then by def of Ric

$$\left(Ric(X) = \right) Ric(X, X) = \sum_{i=1}^{n} R(e_i, X, e_i, X)$$

$$= \sum_{i=2}^{n} R(e_i, e_1, e_i, e_1)$$

$$= \sum_{i=2}^{n} K(\pi_i)$$

where $\pi_i = \mathrm{span}\{e_1, e_i\}$

Def: The scalar curvature $S(x)$ at $x \in M$ is defined by

$$S(x) = \sum_{i,j} R(e_i, e_j, e_i, e_j)$$

where $\{e_1, \cdots, e_n\}$ = o.n. basis of T_xM

i.e. Scalar curvature = sum of all sectional curvatures of planes given by an o.n basis.