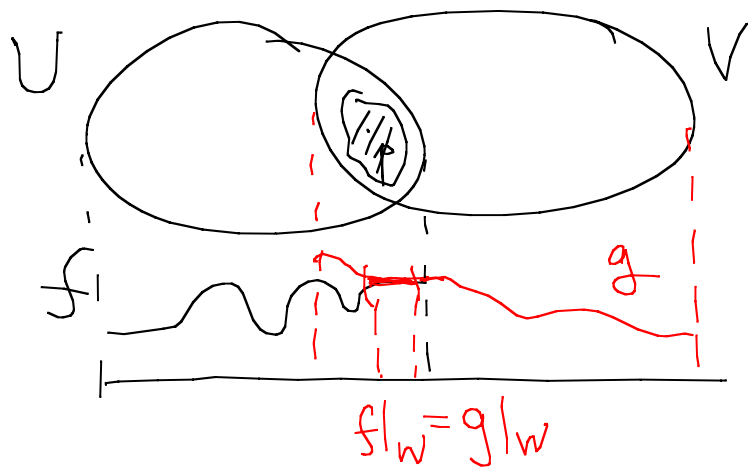


1.5 Tangent vectors as derivations

Let M be a smooth manifold, $p \in M$, consider C^∞ functions defined in a neighborhood of p . Then we can define an

equivalence relation : $f: U \rightarrow \mathbb{R} \sim g: V \rightarrow \mathbb{R}$
($p \in U, p \in V$)

$\Leftrightarrow \exists$ nbd. $W \subset U \cap V$ of p s.t. $f|_W = g|_W$



Def: The equivalence classes for this relation are the germs of C^∞ functions at p . The space of germs of C^∞ functions at p is denoted $\mathcal{L}_p^\infty(M)$.

Similarly, we can define $\mathcal{L}_p^0(M)$, $\mathcal{L}_p^k(M)$ & $\mathcal{L}_p^\omega(M)$ the germs of continuous, C^k , & (real) analytic functions respectively at p .

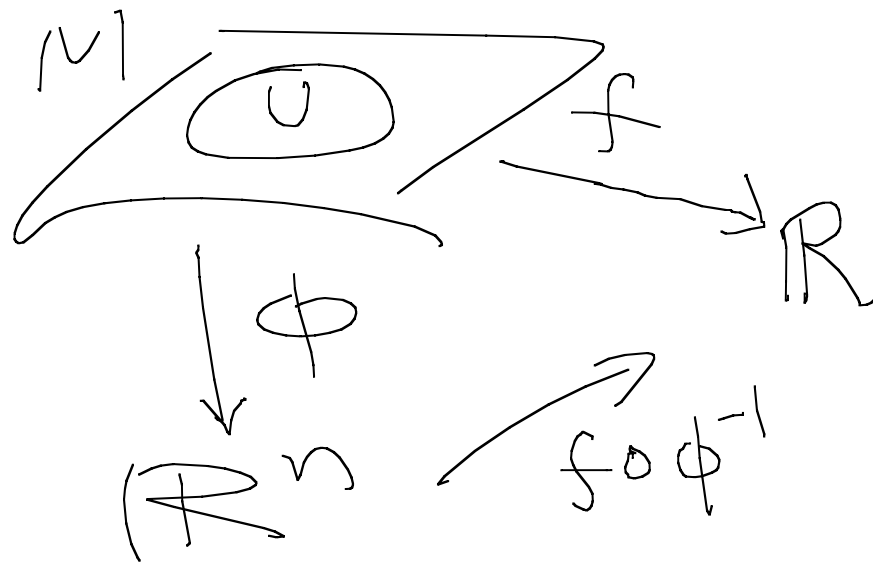
Remarks: • Space of functions has linear structure
(with product structure)

\Rightarrow corresponding space of germs is a vector space (with product structure).

• If M is a C^k manifold ($0 \leq k \leq \infty$)

then $\mathcal{L}_p^k(M) \cong \mathcal{L}_0^k(\mathbb{R}^n)$ (vector space isomorphism)

Pf: germ of $f \leftrightarrow$ germ of $f \circ \phi^{-1}$
for a chart (U, ϕ)



Def: A derivation on $\mathcal{C}_p^k(M)$ is a linear map

$\delta: \mathcal{C}_p^k(M) \rightarrow \mathbb{R}$ such that $\forall f, g \in \mathcal{C}_p^k(M)$

$$\delta(fg) = f(p)\delta(g) + g(p)\delta(f)$$

(Ex.)

(where fg = product of the germs f, g : How to define?)

Notation: We denote the set of derivations on $\mathcal{C}_p^k(M)$ by

$$\mathcal{D}_p^k(M) \simeq \mathcal{D}_p(M) \text{ if } k \text{ is clear.}$$

Thm: Any derivation of $\mathcal{C}_0^\infty(\mathbb{R}^n)$ can be written as

$$\delta(f) = \sum_{j=1}^n \delta(x^j) \frac{\partial f}{\partial x^j}(0)$$

← this f is a function representing the germ f .

Hence $\dim(\mathcal{D}_0^\infty(\mathbb{R}^n)) = n$.

(where x^j = germ of the coordinate function $x^j: \mathbb{R}^n \rightarrow \mathbb{R}$
 $\begin{pmatrix} x^1 \\ \vdots \\ x^n \end{pmatrix} \mapsto x^j$)

Pf: \forall germ $f \in \mathcal{C}_0^\infty(\mathbb{R})$, f is represented by a C^∞ function in a nbd. of 0. We denote this function by f again. Then

$$\begin{aligned} f(x) - f(0) &= \int_0^1 \frac{d}{dt} f(tx) dt \\ &= \int_0^1 \sum_{j=1}^n \frac{\partial f}{\partial x^j}(tx) x^j dt \\ &= \sum_{j=1}^n x^j h_j(x) \end{aligned}$$

where $h_j(x) = \int_0^1 \frac{\partial f}{\partial x_j}(tx) dt \in C^\infty$.

Then $\delta(f) = \delta(f - f(0))$ since $\delta(\text{const.}) = 0$
 (Ex.)

$$= \sum_{j=1}^n \delta(x^j h_j)$$

$$= \sum_{j=1}^n x^j(0) \delta(h_j) + h_j(0) \delta(x^j)$$

$$= \sum_{j=1}^n \delta(x^j) \frac{\partial f}{\partial x^j}(0) \quad \#$$

Lemma $\forall \xi \in T_p M, L_\xi(f) \stackrel{\text{def}}{=} (D_p f)(\xi) \quad \forall f \in C_0^\infty(M)$

Then $L_\xi \in \mathcal{D}_p(M)$

(Pf = Easy Ex)

Thm: $T_p M \rightarrow D_p(M)$ is an isomorphism
 (between vector spaces)

$$\begin{array}{ccc} \in & & \in \\ \xi & \longmapsto & L_\xi \end{array}$$

Pf:

- $\xi \mapsto L_\xi$ is clearly linear
- let (U, ϕ) be a chart for M around p with $\phi(p) = 0$.

Then ξ can be represented by

$$\xi = (U, \phi, v) \quad \text{with } v \in T_0 \mathbb{R}^n \cong \mathbb{R}^n$$

$\Rightarrow \forall C^\infty$ function f in a nbd around p .

$$L_\xi f = D_0(f \circ \phi^{-1})(v)$$

$$= \sum_{j=1}^n v^j \frac{\partial}{\partial x_j} (f \circ \phi^{-1})(0), \quad \text{where } v = (v^1, \dots, v^n)$$

If $\xi \in \ker(\xi \mapsto L_\xi)$, then $\forall f$

$$0 = \sum_{j=1}^n v^{\hat{j}} \frac{\partial}{\partial x^j} (f \circ \phi^{-1})(0)$$

$$\Rightarrow v^{\hat{j}} = 0, \forall j \Rightarrow \xi = 0. \quad \therefore \ker(\xi \mapsto L_\xi) = 0$$

• Finally, $\forall \delta \in \mathcal{D}_p(M) \cong \mathcal{D}_0(\mathbb{R}^n)$, previous lemma

$$\Rightarrow \delta(f) = \sum_{j=1}^n \delta(x^{\hat{j}}) \frac{\partial}{\partial x^j} (f \circ \phi^{-1})(0)$$

$$\therefore \delta = L_\xi \text{ for } \xi = \left[(U, \phi, \begin{pmatrix} \delta(x^1) \\ \vdots \\ \delta(x^n) \end{pmatrix}) \right] \in T_p M$$

$$\Rightarrow \text{Im}(\xi \mapsto L_\xi) = \mathcal{D}_p(M) \quad \times$$

Remark: In particular, we have $\dim T_p M = n$ with basis corresponds to $\left\{ \frac{\partial}{\partial x^i} \Big|_p \right\}$ in local coordinates $\mathcal{D}_0(\mathbb{R}^n)$

(i.e. a germ s.t. $\begin{pmatrix} \delta(x^1) \\ \vdots \\ \delta(x^n) \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} \leftarrow j^{\text{th}} \text{ place}$)

Convention: If (U, ϕ) is a chart around p , and (x^1, \dots, x^n) are the corresponding coordinate functions

$$x^j: U \xrightarrow{\phi} \mathbb{R}^n \xrightarrow{\pi_j} \mathbb{R}.$$

We denote $\left(\frac{\partial}{\partial x^i} \right)_p (f) \stackrel{\text{def}}{=} \frac{\partial (f \circ \phi^{-1})}{\partial x^i} (\phi(p))$

In this notation

$$L_{\xi} = \sum_{j=1}^n v^{\hat{j}} \left(\frac{\partial}{\partial x^{\hat{j}}} \right)_p \quad \text{for } \xi = (U, \phi, v) \in T_p M$$

Hence $\left(\frac{\partial}{\partial x^{\hat{j}}} \right)_p$ can be regarded as a vector in $T_p M$;

$\Rightarrow \frac{\partial}{\partial x^{\hat{j}}}$ is an vector field on $U \subset M$.

If $\bar{x}^1, \dots, \bar{x}^n$ are smooth functions, then

$$\bar{X} = \sum_{\hat{j}=1}^n \bar{x}^{\hat{j}} \frac{\partial}{\partial x^{\hat{j}}} \quad \text{is a vector field on } U.$$

corresponds to $L_{\bar{X}} = C^{\infty}(U) \rightarrow C^{\infty}(U)$ defined by

$$(L_{\bar{X}} f)(p) = \sum_{\hat{j}=1}^n \bar{x}^{\hat{j}}(p) \left(\frac{\partial f}{\partial x^{\hat{j}}} \right)_p$$

Thm: The map $X \mapsto L_X$ is an isomorphism between the vector spaces $\Gamma(TM)$ and $\mathcal{D}(M)$, where $\mathcal{D}(M)$ = set of derivations δ on M which are defined by requiring

(i) $\delta: C^\infty(M) \rightarrow C^\infty(M)$ linear;

(ii) $\delta(fg) = f\delta(g) + g\delta(f)$

(Pf = Omitted)

(Caution: Analog statement for complex manifold is not true. Since we need to use cut-off functions to reduce to coordinate systems.)

Note: If $\delta_1, \delta_2 \in \mathcal{D}(M)$, then $\delta_1 \circ \delta_2 \notin \mathcal{D}(M)$

Lemma: If $\delta_1, \delta_2 \in \mathcal{D}(M)$, then

$$\delta_1 \circ \delta_2 - \delta_2 \circ \delta_1 \in \mathcal{D}(M)$$

Pf (Exercise)

Def: Let X, Y be vector fields on M . Then $[X, Y]$, the bracket of X & Y , is the vector field corresponding

to the derivation $L_X \circ L_Y - L_Y \circ L_X$

(i.e. $L[X, Y] = L_X \circ L_Y - L_Y \circ L_X$)

Local formula for $[X, Y]$

If $X = \sum_{i=1}^n X^i \frac{\partial}{\partial x^i}$, $Y = \sum_{j=1}^n Y^j \frac{\partial}{\partial x^j}$ in some local coordinates

then

$$L_{\underline{X}} f = \sum_i \underline{X}^i \frac{\partial f}{\partial x^i}$$

$$\Rightarrow L_{\underline{Y}} L_{\underline{X}} f = \sum_{i,j} Y^j \underline{X}^i \frac{\partial^2 f}{\partial x^j \partial x^i} + Y^j \frac{\partial \underline{X}^i}{\partial x^j} \frac{\partial f}{\partial x^i}$$

Similar formula for $L_{\underline{X}} L_{\underline{Y}} f$.

$$\Rightarrow (L_{\underline{X}} L_{\underline{Y}} - L_{\underline{Y}} L_{\underline{X}}) f = \sum_i \left(\sum_j Y^j \frac{\partial \underline{X}^i}{\partial x^j} - \underline{X}^j \frac{\partial Y^i}{\partial x^j} \right) \frac{\partial f}{\partial x^i}$$

$$\Rightarrow \left[\underline{X}, \underline{Y} \right] = \sum_i Z^i \frac{\partial}{\partial x^i} \quad \text{with}$$

$$Z^i = \sum_j \left(Y^j \frac{\partial \underline{X}^i}{\partial x^j} - \underline{X}^j \frac{\partial Y^i}{\partial x^j} \right)$$

Lemma (Jacobi identity) For vector fields $\underline{X}, \underline{Y}, \underline{Z}$,

$$[\underline{X}, [\underline{Y}, \underline{Z}]] + [\underline{Y}, [\underline{Z}, \underline{X}]] + [\underline{Z}, [\underline{X}, \underline{Y}]] = 0 \quad (\text{Pf: Ex})$$

1.6 Vector Bundles and Tensors

Def: Let E & B be 2 smooth manifolds and
 $\pi: E \rightarrow B$ be a smooth map.

(π, E, B) is a vector bundle of rank n ,

if

- π is surjective;

- \exists open covering $(U_i)_{i \in \Lambda}$ of B , and

diffeomorphisms $h_i: \pi^{-1}(U_i) \rightarrow U_i \times \mathbb{R}^n$

s.t. $\forall x \in U_i \quad h_i(\pi^{-1}(x)) = \{x\} \times \mathbb{R}^n$

(hence $\pi^{-1}(x)$ can be regarded as a vector space.)

• and such that $\forall \hat{i}, \hat{j} \in \Lambda$, the diffeomorphism

$$h_{\hat{i}} \circ h_{\hat{j}}^{-1} : (U_{\hat{i}} \cap U_{\hat{j}}) \times \mathbb{R}^n \rightarrow (U_{\hat{i}} \cap U_{\hat{j}}) \times \mathbb{R}^n$$

are of the form

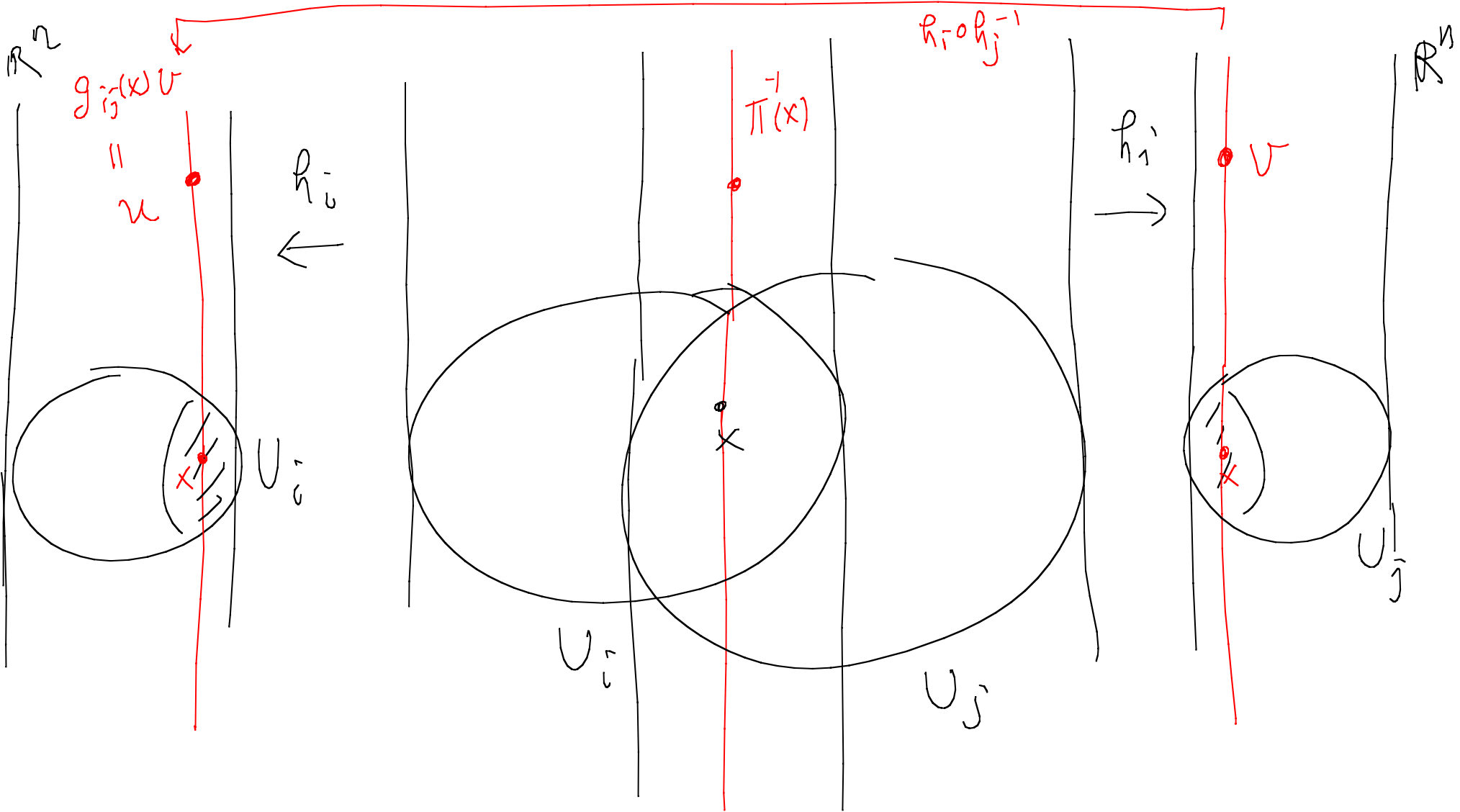
$$h_{\hat{i}} \circ h_{\hat{j}}^{-1}(x, v) = (x, g_{\hat{i}\hat{j}}(x)v)$$

where $g_{\hat{i}\hat{j}} : U_{\hat{i}} \cap U_{\hat{j}} \rightarrow GL(n, \mathbb{R})$

Terminology: $E = \underline{\text{total space}}$, $B = \underline{\text{base}}$.

$\mathbb{R}^n \simeq \pi^{-1}(x) = \underline{\text{fibre}}$

$h_{\hat{i}} = \underline{\text{local trivialization}}$



eg: (Trivial Bundle) : $\pi = M \times \mathbb{R}^n \rightarrow M$
 $(x, v) \mapsto x$

eg: Tangent bundle of $M = TM = \bigsqcup_{p \in M} T_p M$ (Exercise)

Def: (a) A vector bundle of rank n , $\pi: E \rightarrow B$, is trivial

if \exists a diffeomorphism

$$\varphi = E \rightarrow B \times \mathbb{R}^n$$

s.t. $\varphi = \pi^{-1}(x) \rightarrow \{x\} \times \mathbb{R}^n$ is a

vector space isomorphism.

(b) A (global) section of the bundle is a smooth

map $s: B \rightarrow E$ s.t. $\pi \circ s = \text{id}$

$$\begin{array}{ccc} & & E \\ & & \pi \downarrow \uparrow s \\ & & B \end{array}$$

eg: vector field $X \in \Gamma(M) (= \Gamma(TM))$ is a section of the tangent bundle TM ,

Tensor product

Def: Let E, F be 2 finite dimensional vector spaces, then $E \otimes F$, the tensor product of E & F , is defined as the vector space, unique up to isomorphism, such that \forall vector space G ,

$$\begin{array}{ccc} L(E \otimes F, G) & \xrightarrow{\cong} & L_2(E \times F, G) \\ \text{(linear transformations)} & & \text{(bilinear maps from)} \\ \text{(from } E \otimes F \text{ to } G) & & E \times F, G \end{array}$$

Remark: \exists a bilinear map $\otimes: E \times F \rightarrow E \otimes F$ such that if $\{e_i\}$ = basis of E &

$\{f_j\}$ = basis of F ,

then $\{e_i \otimes f_j\}_{i,j}$ is a basis of $E \otimes F$

(Hence for $u = a^i e_i \in E$, $v = b^{\hat{j}} f_j \in F$, then)

$$u \otimes v = a^i b^{\hat{j}} e_i \otimes f_j$$

Facts: (1) If $E^* = \text{dual of } E = L(E, \mathbb{R})$

$F^* = \text{dual of } F$

then $E^* \otimes F^* \simeq L_2(E \times F, \mathbb{R})$

$\simeq L(E \otimes F, \mathbb{R}) = (E \otimes F)^*$

(by \downarrow
 $\alpha \otimes \beta \longmapsto \alpha \otimes \beta(u \otimes v) = \alpha(u)\beta(v)$.)

(2) If $\alpha \in L(E, E')$ & $\beta \in L(F, F')$

(E, E', F, F' are finite dim'l vector spaces)

then one can define

$$\alpha \otimes \beta \in L(E \otimes F, E' \otimes F')$$

by $(\alpha \otimes \beta)(u \otimes v) \stackrel{\text{def}}{=} \alpha(u) \otimes \beta(v)$

[If $E' = \mathbb{R} = F'$, then $E' \otimes F' = \mathbb{R}$
 $\Rightarrow L(E \otimes F, E' \otimes F') \cong L(E \otimes F, \mathbb{R})$ as in (1)]

(3) Given a vector bundle E (with fibers $E_x, x \in M$),

one can define the vector bundle E^* , $\otimes^P E$

(with fibers E_x^* , and $\otimes^P E_x$ respectively)

(4) Given 2 vector bundles E, F (with fibers E_x, F_x)
with the same base manifold M , we can define
the vector bundle $E \otimes F$ over M with fiber $E_x \otimes F_x$.

eg: Starting from TM , we can define the cotangent bundle
 T^*M of M , and the (p, q) -tensor bundle

$$\left(\otimes^p TM \right) \otimes \left(\otimes^q T^*M \right) \text{ of } M$$

Def: A (p, q) -tensor (field), or more precisely
 p times contravariant & q times covariant tensor,
on M is a smooth section of the bundle

$$(\otimes^p TM) \otimes (\otimes^q T^*M).$$

Note: For $f: M \rightarrow \mathbb{R}$ smooth, we can define

$$df \in \Gamma(T^*M) \text{ by } df(X) = L_X f, \quad \forall X \in \Gamma(TM)$$

$$(Tf = Df) \quad (= Xf)$$

Then $\{dx^{\hat{j}}\}_{\hat{j}=1}^n$ is a dual basis to $\{\frac{\partial}{\partial x^{\hat{i}}}\}_{\hat{i}=1}^n$

$$\left(dx^{\hat{j}} \left(\frac{\partial}{\partial x^{\hat{i}}} \right) = \frac{\partial}{\partial x^{\hat{i}}} (x^{\hat{j}}) = \delta_{\hat{i}}^{\hat{j}} \right)$$

at each point in a coordinate system with coordinate functions x^1, \dots, x^n .

Therefore

$$\left\{ \frac{\partial}{\partial x^{\hat{j}_1}} \otimes \dots \otimes \frac{\partial}{\partial x^{\hat{j}_p}} \otimes dx^{i_1} \otimes \dots \otimes dx^{i_q} \right\}$$

forms a basis for $(\otimes^p TM) \otimes (\otimes^q T^*M)$.

\Rightarrow in coordinates, a (p, q) -tensor (field) can be written as

$$T = T_{\substack{\hat{j}_1 \dots \hat{j}_p \\ i_1 \dots i_q}} \frac{\partial}{\partial x^{\hat{j}_1}} \otimes \dots \otimes \frac{\partial}{\partial x^{\hat{j}_p}} \otimes dx^{i_1} \otimes \dots \otimes dx^{i_q}$$

1.7 Partitions of unity

Recall that all manifolds in this course are supposed to have the property that "partitions of unity" is

always possible.

That is :

for any $\{U_i\}_{i \in \Lambda}$ = open cover of M ,

\exists locally finite open cover $\{V_k\}_{k \in \Lambda'}$ and

a family $\{\varphi_k\}_{k \in \Lambda'}$ of real smooth functions

on M such that

- $\{V_k\}_{k \in \Lambda'}$ is subordinate to $\{U_i\}_{i \in \Lambda}$
(i.e. each $V_k \subset U_i$ for some i)
- $\text{supp } \varphi_k \subset V_k$, $\varphi_k \geq 0$, $\sum_{k \in \Lambda'} \varphi_k(x) = 1$, $\forall x \in M$.

Here $\{V_k\}_{k \in \Lambda'}$ being locally finite means

$\forall x \in M, \exists$ open nbhd W of x such that

$W \cap V_k = \emptyset$ except finite many k 's.