MATH5011 Real Analysis I

Exercise 8 Suggested Solution

Standard notations are in force. Those with *,taken from [R], are optional.

(1) Let $f, g \in L^p(\mu)$, $1 < p < \infty$. Show that the function

$$\Phi(t) = \int_X |f + tg|^p \, d\mu$$

is differentiable at $t = 0$ and

$$\Phi'(0) = p \int_X |f|^{p-2} f g \, d\mu.$$

Hint: Use the convexity of $t \mapsto |f + tg|^p$ to get

$$|f + tg|^p - |f|^p \le t(|f + g|^p - |f|^p), \quad t > 0$$

and a similar estimate for $t < 0$.

Solution: Recall that for any convex function φ defined on $[0, 1]$, one has the elementary inequality

$$\frac{\varphi(t) - \varphi(0)}{t - 0} \le \frac{\varphi(1) - \varphi(0)}{1 - 0}, \quad \forall t \in (0, 1),$$

which could be deduced from the definition of convexity. For $p > 1, x \in X$, the function $\varphi(t) = |f(x) + tg(x)|^p$ is convex, differentiable and

$$\lim_{t \to 0} \frac{|f(x) + tg(x)|^p - |f|^p(x)}{t} = p|f|^{p-2}(x)(f(x)g(x)),$$

whenever $f(x)$ and $g(x)$ are finite. Applying the inequality above to this particular convex function, We have

$$\frac{1}{t}\{|f+tg|^p - |f|^p\} \le |f+g|^p - |f|^p,\ \forall t \in (0,1).$$

By replacing t with $-t$, we obtain a similar inequality

$$|f|^p - |f-g|^p \le \frac{1}{t}\{|f+tg|^p - |f|^p\},\ \forall t \in (-1,0).$$

Now the desired result follows from an application of Lebesgue's dominated convergence theorem.

(2) Suppose f is a measurable function on X, μ is a positive measure on X, and

$$\varphi(p) = \int_X |f|^p \, d\mu = \|f\|_p^p \quad (0 < p < \infty).$$

Let $E = \{p : \varphi(p) < \infty\}$. Assume $\|f\|_\infty > 0$.

(a) If $r < p < s$, $r \in E$, and $s \in E$, prove that $p \in E$.

(b) Prove that $\log \varphi$ is convex in the interior of E and that φ is continuous on E.

(c) By (a), E is connected. Is E necessarily open? Closed? Can E consist of a single point? Can E be any connected subset of $(0,\infty)$?

(d) If $r < p < s$, prove that $\|f\|_p \le \max(\|f\|_r, \|f\|_s)$. Show that this implies the inclusion $L^r(\mu) \cap L^s(\mu) \subset L^p(\mu)$.

(e) Assume that $\|f\|_r < \infty$ for some $r < \infty$ and prove that

$$\|f\|_p \to \|f\|_\infty \qquad \text{as } p \to \infty.$$

Solution:

(a) Write $p = \lambda r + (1-\lambda)s$ for $0 < \lambda < 1$. By Hölder's inequality,

$$\int_X |f|^p \, d\mu = \int_X |f|^{\lambda r} |f|(1-\lambda)s \, d\mu \le \left(\int_X |f|^r \, d\mu \right)^{\lambda} \left(\int_X |f|^s \, d\mu \right)^{1-\lambda},$$

which shows that φ is finite on $[r, z]$.

(b) Rewrite the inequality above as

$$\varphi(\lambda r + (1-\lambda)s) \le \varphi(r)^{\lambda} \cdot \varphi(s)^{1-\lambda}, \quad (0 < \lambda < 1).$$

It is also true for $\lambda = 0, 1$. Hence for all $\lambda \in [0, 1]$,

$$\log \varphi(\lambda r + (1-\lambda)s) \le \lambda \log \varphi(r) + (1-\lambda) \log \varphi(s).$$

since log is increasing. Thus $\log \varphi(p)$ is convex on $[r, s]$. Hence $\varphi(x)$ is continuous in the interior of E. It follows form monotonicity applying to $\chi_{|f|>1} f$ and $\chi_{|f|\le 1} f$ that $\varphi(x)$ is also continuous on ∂E.

(c) Let $X = (0, \infty)$ with the Lebesgue measure. E can be any connected subset of $(0, \infty)$. The basic functions to consider are of the form x^k and $x^k |\log x|^m$ near $x = 0$ and $x = \infty$. Define

$$\begin{aligned} g_k(x) &= x^k \chi_{(0,1/2]}(x), \\ h_k(x) &= x^k \chi_{(2,\infty)}(x), \\ g_{k,m}(x) &= x^k |\log x|^m \chi_{(0,1/2]}(x), \\ h_{k,m}(x) &= x^k |\log x|^m \chi_{(2,\infty)}(x), \end{aligned}$$

It is easy to see that $\int_X g_k dx < \infty$ iff $k > -1$ and $\int_X h_k dx < \infty$ iff $k < -1$. Since $|\log x| \le C_\epsilon e^{-\epsilon}$ for $0 \le x \le 1$ and all $\epsilon > 0$, $\int_X g_{k,m} dx$ is finite for $k > -1$ and infinite for $k > -1$. For $k = -1$, direct computations by substituting $u = \log x$ yield

$$\int_X g_{k,m} dx = \int_0^{1/2} x^{-1} |\log x|^m dx = \int_{\log 2}^{\infty} u^m du,$$

which is finite iff $m < -1$. Similarly, one can show $\int_X h_{k,m} dx$ is finite for $k > -1$ and infinite for $k > -1$. If $k = -1$, the integral is finite if and only if $m < -1$. Note that $g_k^p = g_{pk}, g_{k,m}^p = g_{pk,pm}$ and similarly for h.

Now for $f = g_{-1,-2} + h_{-1,-2}$, one has $E = 1$. For $E = \emptyset$, take $f = g_{-1} + h_{-1}$. To get $E = (0, \infty)$, one may take $f = e^{-|x|}$. For $E = [1, p)$, take $f = g_{-1/p} + h_{-1,-2}$. Similarly it is easy to see that E can be any connected subset of $(0, \infty)$ for choosing f properly.

(d) The inequality in (a) implies $\|f\|_p \leq \max(\|f\|_r, \|f\|_s)$. Obviously, if $\|f\|_r < \infty$ and $\|f\|_s < \infty$, then $\|f\|_p < \infty$. Thus $L^r(\mu) \cap L^s(\mu) \subset L^p(\mu)$.

(e) Denote $E_a := \{x : a \leq |f(x)|\}$ for every $0 < a < \|f\|_\infty$, then $0 < \mu(E_a) < \infty$. ($\|f\|_r < \infty$ implies $\mu(E_a) < \infty$.) Thus

$$\|f\|_p = \left(\int_X |f|^p \, d\mu \right)^{1/p} \geq \left(\int_{E_a} |f|^p \, d\mu \right)^{1/p} \geq a(\mu(E_a))^{1/p},$$

which implies $\varliminf_{p \to \infty} \|f\|_p \geq a$. Since a is arbitrary, we have $\varliminf_{p \to \infty} \|f\|_p \geq \|f\|_\infty$.

On the other hand, for $p > r$,

$$\|f\|_p = \left(\int_X |f|^{p-r} |f| r \, d\mu \right)^{1/p} \leq \|f\|_r^{r/p} \|f\|_\infty^{1-r/p},$$

which implies $\varlimsup_{p \to \infty} \|f\|_p \leq \|f\|_\infty$. In conclusion, we have

$$\lim_{p \to \infty} \|f\|_\infty = \|f\|_\infty.$$

(3) Assume, in addition to the hypothesis of the last problem, that

$$\mu(X) = 1.$$

(a) Prove that $\|f\|_r \leq \|f\|_s$ if $0 < r < s \leq \infty$.

(b) Under what conditions does it happen that $0 < r < s \leq \infty$ and $\|f\|_r = \|f\|_s < \infty$?

(c) Prove that $L^r(\mu) \supset L^s(\mu)$ if $0 < r < s$. Under what conditions do these two spaces contain the same functions?

(d) Assume that $\|f\|_r < \infty$ for some $r > 0$, and prove that

$$\lim_{p \to 0} \|f\|_p = \exp\left\{ \int_X \log|f|\, d\mu \right\}$$

if $\exp\{-\infty\}$ is defined to be 0.

Solution:

(a) If $s < \infty$, the conclusion from from Hölder's inequality,

$$\int_X |f|^r\, d\mu \leq \left(\int_X |f|^s\, d\mu \right)^{r/s} \left(\int_X 1\, d\mu \right)^{1-r/s} = \|f\|_s^r.$$

If $s = \infty$, the desired result follows from

$$\|f\|_r \leq \|f\|_\infty \left(\int_X 1\, d\mu \right)^{1/r} = \|f\|_\infty.$$

(b) From the equality sign characterization in the Hölder inequality it is easy to see that $\|f\|_r = \|f\|_s < \infty$ if and only if $|f| = \|f\|_\infty < \infty$ a.e..

(c) We claim that under the condition $\mu(X) < \infty$, $L^r(\mu) = L^s(\mu)$ for $0 < r < s \leq \infty$ if and only if the following property (call it L) holds:
There exists $\varepsilon_0 > 0$ such that for any measurable set $E \subset X$ with $\mu(E) > 0$ we have $\mu(E) > \varepsilon_0$.

In fact, if Property L holds, let $f \in L^r(\mu)$ and denote $E_n := \{x : |f| \geq n\}$. Then there exists $n_0 \in \mathbb{N}$ such that $\mu(E_{n_0}) = 0$ and thus $f \in L^\infty(\mu)$. Otherwise for all n, $\mu(E_n) > 0$. Thus $\mu(\{x : |f(x)| = \infty\}) \geq \lim_{n\to\infty} \mu(E_n) \geq \varepsilon_0$ and then $\|f\|_r = \infty$, a contradiction.

Conversely, suppose there is a sequence of measurable sets $\{E_n\}$ with $0 < \mu(E_n) < 3^{-n}$. Without loss of generality, E_n are mutually disjoint. Denote $a_n := \mu(E_n)$ and define

$$f = \begin{cases} \sum_{n=1}^{\infty} a_n^{-1/s} \chi_{E_n}, & \text{if } s < \infty, \\ \sum_{n=1}^{\infty} a_n^{-\frac{1}{2r}} \chi_{E_n}, & \text{if } s = \infty. \end{cases}$$

Then $f \in L^r$ but $f \notin L^s$. The proof is completed.

(d) Note $x - 1 - \log x \geq 0$ on $[0, \infty)$ implies that

$$\int_{\{|f|>1\}} \log |f| d\mu < \infty.$$

If $\mu(\{|f| = 0\}) > 0$, it suffices to proves the equality by showing $\lim_{p\to 0} \|f\|_p = 0$. There is a small $s > 1$, with s' be its conjugate s.t.

$$\begin{aligned} \|f\|_p &= \{\int_X |f|^p \chi_{\{|f|>0\}} d\mu\}^{\frac{1}{p}} \\ &\leq (\mu\{|f| > 0\})^{\frac{1}{s'p}} \|f\|_{sp} \text{ by Hölder inequality} \\ &\leq (\mu\{|f| > 0\})^{\frac{1}{s'p}} \|f\|_r \to 0 \text{ as } p \to 0 \end{aligned}$$

We may suppose $\infty > |f| > 0$ a.e. By Jensen's inequality, we have

$$\log \|f\|_p = \frac{1}{p} \log \int_X |f|^p \, d\mu \geq \frac{1}{p} \int_X \log |f|^p \, d\mu = \int_X \log |f| \, d\mu.$$

On the other hand, $x - 1 - \log x \geq 0$ on $[0, \infty)$ implies $\dfrac{\|f\|_p^p - 1}{p} \geq \log \|f\|_p$. Thus

$$\int_X \log |f| \, d\mu \leq \log \|f\|_p \leq \int_X \frac{|f|^p - 1}{p} \, d\mu$$

since $\mu(X) = 1$. Note that by convexity of the map $p \mapsto |f|^p$ we have $\dfrac{|f|^p - 1}{p}$ is increasing in p, which implies $\dfrac{|f|^p - 1}{p} \leq \dfrac{|f|^r - 1}{r} \in L^1(\mu)$ and $\displaystyle\lim_{p \to 0} \frac{|f|^p - 1}{p} = \log |f|$. By Lebesgue's dominated convergence theorem for $|f| > 1$ and monotone convergence theorem for $|f| < 1$, we have

$$\lim_{p \to 0} \int_X \frac{|f|^p - 1}{p} \, d\mu = \lim_{p \to 0} \int_{\{|f| \geq 1\}} \frac{|f|^p - 1}{p} \, d\mu + \lim_{p \to 0} \int_{\{|f| < 1\}} \frac{|f|^p - 1}{p} \, d\mu = \int_X \log |f| \, d\mu.$$

Thus by sandwich rule

$$\lim_{p \to 0} \|f\|_p = \exp \left\{ \int_X \log |f| \, d\mu \right\}$$

(4) For some measures, the relation $r < s$ implies $L^r(\mu) \subset L^s(\mu)$; for others, the inclusion is reversed; and there are some for which $L^r(\mu)$ does not contain $L^s(\mu)$ is $r \neq s$. Give examples of these situations, and find conditions on μ under which these situations will occur.

Solution:

First, we give examples of these situations:

(a) For $X = [0, 1]$ with usual Lebesgue measure, we have $L^r(\mu) \supset L^s(\mu)$ if $r < s$.

(b) For $X = \mathbb{N}$ with counting measure, we have $L^r(\mu) \subset L^s(\mu)$ if $r < s$.

(c) For $X = \mathbb{R}$ with usual Lebesgue measure, we have $L^r(\mu) \neq L^s(\mu)$ if $r \neq s$.

Second, we give simple conditions on μ under which these situations occur correspondingly:

(a) $\mu(X) < \infty$.

(b) Property L in 6(c) holds.

(c) $\mu(X) = \infty$ and Property L in 6(c) fails to hold.

(5) * Suppose $\mu(\Omega) = 1$, and suppose f and g are positive measurable functions on Ω such that $fg \geq 1$. Prove that

$$\int_\Omega f\, d\mu \cdot \int_\Omega g\, d\mu \geq 1.$$

Solution: Since $fg \geq 1$, we have $\sqrt{fg} \geq 1$ and so by Hölder's inequality,

$$1 \leq \int_\Omega \sqrt{f}\sqrt{g}\, d\mu \leq \left(\int_\Omega f\, d\mu\right)^{1/2} \left(\int_\Omega g\, d\mu\right)^{1/2} = \left(\int_\Omega f\, d\mu \cdot \int_\Omega g\, d\mu\right)^{1/2}.$$

(6) * Suppose $\mu(\Omega) = 1$ and $h : \Omega \to [0, \infty]$ is measurable. If

$$A = \int_\Omega h\, d\mu,$$

prove that

$$\sqrt{1 + A^2} \leq \int_\Omega \sqrt{1 + h^2}\, d\mu \leq 1 + A.$$

If μ is Lebesgue measure on $[0, 1]$ and if h is continuous, $h = f'$, the above inequalities have a simple geometric interpretation. From this, conjecture (for general Ω) under what conditions on h equality can hold in either of the above inequalities, and prove your conjecture.

Solution: The function $\phi(x) = \sqrt{1 + x^2}$ is convex since its second derivative is always positive. Hence the first inequality follows from Jensen's inequality. The second equality follows from $|\Omega| = 1$ and $\sqrt{1 + x^2} \leq 1 + x$ for all $x \geq 0$.

In the case that $\Omega = [0,1]$ with μ the Lebesgue measure and $h = f'$ is continuous, then $\int_0^1 \sqrt{1+(f')^2}dx$ is the arc length of the graph of f. Then $A = f(1)-f(0)$. The first inequality means that the straight line is the shortest path while the second inequality means the longest path is the segment from $(0, f(0))$ to $(1, f(0))$ and then going up until $(1, f(1))$.

The intuition from this suggests that the second inequality is equality if and only if $h = 0, a.e.$, and the first inequality is equality if and only if $h = A, a.e$. The first claim is clear since $\sqrt{1+x^2} = 1+x$ iff $x = 0$.If $h = A, a.e$, then trivially the first inequality holds. Conversely if the first inequality holds, it follows from an examination of the proof of Jensen's inequality that $\phi(A) = \phi(h(x)), a.e.$, so $h = A, a.e.$ since ϕ is injective on $[0, \infty)$.

(7) * Suppose $1 < p < \infty$, $f \in L^p = L^p((0,\infty))$, relative to Lebesgue measure, and

$$F(x) = \frac{1}{x}\int_0^x f(t)\,dt \quad (0 < x < \infty).$$

(a) Prove Hardy's inequality

$$\|F\|_p \le \frac{p}{p-1}\|f\|_p$$

which shows that the mapping $f \to F$ carries L^p into L^p.

(b) Prove that equality holds only if $f = 0$ a.e.

(c) Prove that the constant $\dfrac{p}{p-1}$ cannot be replaced by a smaller one.

(d) If $f > 0$ and $f \in L^1$, prove that $F \notin L^1$.

Suggestions: (a) Assume first that $f \ge 0$ and $f \in C_c((0,\infty))$. Integration by parts gives

$$\int_0^\infty F^p(x)\,dx = -p\int_0^\infty F^{p-1}(x)xF'(x)\,dx.$$

Note that $xF' = f - F$, and apply Hölder's inequality to $\displaystyle\int F^{p-1}f$. Then derive the general case.

(c) Take $f(x) = x^{-1/p}$ on $[1, A]$, $f(x) = 0$ elsewhere, for large A. See also Exercise 14, Chap. 8 in [R].

Solution: In fact we can show the inequality

$$\int_0^\infty |F|^p\, dx \leq \frac{p}{p-1} \int_0^\infty |f||F|^{p-1}\, dx.$$

(a) $\vdash \|F\|_p \leq \dfrac{p}{p-1}\|f\|_p, f \in \mathcal{L}^p(0,\infty), p \in (1,\infty)$

Let $f \in C_c(0,\infty), f \geq 0,$ first

$$\begin{aligned}\int_0^\infty F^p(x)dx &= xF^p(x)\big|_0^\infty - p\int_0^\infty F^{p-1}F'xdx \\ &= 0 - p\int_0^\infty F^{p-1}(f-F)dx,\end{aligned}$$

so

$$\int_0^\infty F^p(x)dx = \frac{p}{p-1}\int_0^\infty F^{p-1}fdx. \tag{1}$$

By Hölder's inequality,

$$\int_0^\infty F^p(x)dx \leq \frac{p}{p-1}\Big\{\int_0^\infty F^p(x)dx\Big\}^{\frac{1}{q}}\|f\|_p$$

and (a) holds.

Now, for $f \in C_c(0,\infty)$, use

$$|F| \leq \frac{1}{x}\int_0^x |f|$$

to get the same inequality.

Finally, for $f \in L^p(0,\infty)$, let $f_n \in C_c(0,\infty)$, $f_n \to f$ in L^p. Use an approximation argument to show $\{F_n\}$ is Cauchy and tends to F in $\mathcal{L}^p$ norm.

(b) $\vdash$ " $=$ " hold iff $f = 0$ a.e.

Let f satisfy

$$||F||_p = \frac{p}{p-1}\|f\|_p.$$

If f changes sign,

$$\widetilde{F}(x) = \frac{1}{x}\int_0^x |f|dt$$

$$\|\widetilde{F}\|_p > \|F\|_p = \frac{p}{p-1} = \||f|\|_p$$

Impossible. Therefore $f \geq 0$ say. By an approximation argument one can show that (1) holds for $f \geq 0$, $f \in L^p$. Following the proof in (a) one see by the equality condition in Hölder's inequality that $f^p =$ const $(F^{p-1})^q$, which implies there exists some positive constant c such that $F(x) = cf(x)$ a.e. Express this as an ODE for F and and solve it to get $f \equiv 0$ if $f \in L^p(0,\infty)$.

(c) Define

$$f(x) = \begin{cases} x^{-1/p}, & \text{if } x \in [1, A], \\ 0, & \text{otherwise.} \end{cases}$$

Then $\|f\|_p = (\log A)^{1/p}$ and

$$F(x) = \begin{cases} 0, & \text{if } x \in (0,1), \\ \frac{p}{p-1}\left(x^{-\frac{1}{p}} - x^{-1}\right), & \text{if } x \in [1, A], \\ \frac{p}{p-1}\left(A^{1-\frac{1}{p}} - 1\right)x^{-1}, & \text{if } x \in (A, \infty). \end{cases}$$

Then $\|F\|_p^p = I_1 + I_2$, where

$$\begin{aligned}
I_1 &= \int_1^A \left(\frac{p}{p-1} \left(x^{-\frac{1}{p}} - x^{-1} \right) \right)^p \, dx \\
&= \left(\frac{p}{p-1} \right)^p \int_1^A \left(x^{-\frac{1}{p}} - x^{-1} \right)^p \, dx \\
I_2 &= \int_A^\infty \left(\frac{p}{p-1} \left(A^{1-\frac{1}{p}} - 1 \right) x^{-1} \right)^p \, dx \\
&= \frac{p^p}{(p-1)^{p+1}} \left(1 - A^{\frac{1}{p}-1} \right)^p \, dx.
\end{aligned}$$

Suppose on the contrary that the constant $\dfrac{p}{p-1}$ can be replaced by $\dfrac{\gamma p}{p-1}$ for some $\gamma \in (0,1)$. Then there exists $\delta \in (\gamma, 1)$. Note that there exists $A_0 > 1$ such that for $x > A_0$, $x^{-\frac{1}{p}} - x^{-1} > \delta x^{-\frac{1}{p}}$. Then for sufficiently large $A > A_0$,

$$\begin{aligned}
I_1 &> \frac{\delta p}{p-1} \int_{A_0}^A x^{-1} \, dx \\
&= \frac{\delta p}{p-1} (\log A - \log A_0) \\
&> \frac{\gamma p}{p-1} \log A \\
&= \frac{\gamma p}{p-1} \|f\|_p^p.
\end{aligned}$$

This implies $\|F\|_p > \dfrac{\gamma p}{p-1} \|p\|_f$ if A is sufficiently large. Contradiction arises.

(d) Since $f > 0$ on $(0,\infty)$, there exists $x_0 > 0$ such that $c_0 := \displaystyle\int_0^{x_0} f(t)\, dt$. Then

$$\int_{x_0}^\infty F(x)\, dx = \int_{x_0}^\infty \frac{1}{x} \int_0^x f(t)\, dt dx \geq \int_{x_0}^\infty \frac{1}{x} \int_0^{x_0} f \, dt dx \geq \int_{x_0}^\infty \frac{c_0}{x} \, dx = \infty,$$

showing that $F \notin L^1$.

(8) Consider $L^p(\mathbb{R}^n)$ with the Lebesgue measure where $0 < p < \infty$. Show that

$\|f+g\|_p \le \|f\|_p + \|g\|_p$ holds $\forall f, g$ implies that $p \ge 1$. Hint: For $0 < p < 1$, $x^p + y^p \ge (x+y)^p$.

Solution: Recall that in fact we have, for $x, y \ge 0$,

$$\begin{cases} x^p + y^p & \ge & (x+y)^p, \quad 0 < p < 1, \\ x^p + y^p & = & (x+y)^p, \quad p = 1, \\ x^p + y^p & \le & (x+y)^p, \quad 1 < p < \infty. \end{cases}$$

Pick any $a, b \ge 0$ and define $f, g \in L^p(\mathbb{R}^n)$ by

$$f(x) = \begin{cases} a, & x \in [0,1]^n, \\ 0, & \text{otherwise.} \end{cases}$$

and

$$g(x) = \begin{cases} b, & x \in [2,3]^n, \\ 0, & \text{otherwise.} \end{cases}$$

Simple calculations show that $\|f\|_p = a$, $\|g\|_p = b$ and $\|f+g\|_p = (a^p + b^p)^{1/p}$. Now the hypothesis implies $a^p + b^p \ge (a+b)^p$. Hence, $p \ge 1$.

(9) Consider $L^p(\mu)$, $0 < p < 1$. Then $\dfrac{1}{q} + \dfrac{1}{p} = 1$, $q < 0$.

(a) Prove that $\|fg\|_1 \ge \|f\|_p \|g\|_q$.

(b) For $f, g \ge 0$, $\|f+g\|_p \ge \|f\|_p + \|g\|_p$.

(c) $d(f,g) \overset{\text{def}}{=} \|f-g\|_p^p$ defines a metric on $L^p(\mu)$.

Solution:

(a) Assume that $g > 0$ everywhere first. Applying Hölder's inequality with conjugate exponents $\widetilde{p} = \dfrac{1}{p}$ and $\widetilde{q} = \dfrac{1}{1-p} = \dfrac{\widetilde{p}}{\widetilde{p}-1}$, we have

$$
\begin{aligned}
\left\||f|^p\right\|_1 &= \left\||fg|^{1/\widetilde{p}}|g|^{-1/\widetilde{p}}\right\|_1 \\
&\le \left\||fg|^{1/p}\right\|_{\widetilde{p}} \left\||g|^{-1/p}\right\|_{\widetilde{q}} \\
&= \|fg\|_1^{1/\widetilde{p}} \left\||g|^{-1/(\widetilde{p}-1)}\right\|_1^{(\widetilde{p}-1)/\widetilde{p}} \\
&= \|fg\|_1^{p} \left\||g|^{-p/(1-p)}\right\|_1^{1-p}, \text{ so} \\
\left\||f|^p\right\|_1^{1/p} &\le \|fg\|_1 \left\||g|^{-p/(1-p)}\right\|_1^{1/p-1} \\
&= \|fg\|_1 \left\||g|^{q}\right\|_1^{-1/q}, \text{ or} \\
\|f\|_p &\le \|fg\|_1 \|g\|_q^{-1}, \text{ that is} \\
\|fg\|_1 &\ge \|f\|_p \|g\|_q .
\end{aligned}
$$

For a general $g \geq 0$, apply the result to $g_\varepsilon = g + \varepsilon$ first and then let g_ε tend to g.

(b) Without loss of generality, we can assume $\|f+g\|_p \neq 0$. Using part (a), we have

$$
\begin{aligned}
\|f+g\|_p^p &= \int (f+g)^p \, d\mu \\
&= \int f(f+g)^{p-1} \, d\mu + \int g(f+g)^{p-1} \, d\mu \\
&\ge (\|f\|_p + \|g\|_p) \left(\int (f+g)^{(p-1)\left(\frac{p}{p-1}\right)} \, d\mu \right)^{1-\frac{1}{p}} \\
&= (\|f\|_p + \|g\|_p) \|f+g\|_p^{p-1}, \text{ so} \\
\|f+g\|_p &\ge \|f\|_p + \|g\|_p .
\end{aligned}
$$

(c) The fact that for $x, y \geq 0$ and $0 < p < 1$,

$$(x+y)^p \le x^p + y^p$$

implies

$$\int |f+g|^p \, d\mu \le \int |f|^p \, d\mu + \int |g|^p \, d\mu.$$

Hence, $d(f,g) \stackrel{\text{def}}{=} \|f-g\|_p^p$ defines a metric on $L^p(\mu)$.

(10) Give a proof of the separability of $L^p(\mathbb{R}^n)$, $1 \le p < \infty$, without using Weierstrass approximation theorem.

Suggestion: Cover $\mathbb{R}^n$ with many cubes and consider the combinations $s = \sum \alpha_j \chi_{C_j}$ where C_j are the cubes and $\alpha_j \in \mathbb{Q}$.

Solution: See the proof of Problem 11(b).

(11) (a) Let X_1 be a subset of the metric space (X,d). Show that (X_1,d) is separable if (X,d) is separable.

(b) Let $E \subset \mathbb{R}^n$ be Lebesgue measurable and consider $L^p(E)$, $1 \le p < \infty$, where the measure is understood to be the restriction of $\mathcal{L}^n$ on E. Is it separable?

Solution:

(a) let $\{x_i\}$ be a countable dense subset of the metric space, fix natural numbers i,j we pick an element from $X_1 \cap B(x_i, 1/j)$(Ball centre at x_i with radius be $1/j$) if it is non-empty. The resulting set is obviously a countable dense subset in X_1

(b) By treating $L^p(E)$ as a subset of $L^p(\mathbb{R}^n)$, it suffices to prove that the later space is separable. Cover $\mathbb{R}^n$ with many cubes and consider the combinations $s = \sum \alpha_j \chi_{C_j}$ where C_j are the cubes and $\alpha_j \in \mathbb{Q}$. $\exists s_m$ such that $s_m \to f$ in L^p-norm, where each s_m has the form as s and hence $\{s_m\}$ is countable.

Step 1. $f \in C_c(\mathbb{R}^n)$, $f \ge 0$.

For each $m = 1, 2, \ldots$, cover $\mathbb{R}^n$ by cubes $C_{m,j}$ of side length

2^{-m}. Define $s_m : \mathbb{R}^n \to \mathbb{R}$ by

$$s_m(x) = \sum_j \alpha_j \chi_{C_{m,j}},$$

where $\alpha_j = 2^{-m} \left\lfloor 2^m \inf_{C_{m,j}} f \right\rfloor$. Now, we have $0 \leq s_m \nearrow f$, or $f - s_m \searrow 0$, thus $(f - s_m)^p \searrow 0$. Since $0 \leq f - s_m \leq f$, we can apply Lebesgue dominated convergence theorem to obtain

$$\lim_{m\to\infty} \|f - s_m\|_p = \left(\int_{\mathbb{R}^n} \lim_{m\to\infty} (f - s_m)^p \, d\mathcal{L}^n \right)^{\frac{1}{p}} = 0.$$

Step 2. $f \in C_c(\mathbb{R}^n)$.

Write $f = f_+ - f_-$. Use $s_m^+ \nearrow f_+$ and $s_m^- \nearrow f_-$ in L^p-norm, as in Step 1. Then

$$\left\| f - (s_m^+ - s_m^-) \right\|_p \leq \left\| f_+ - s_m^+ \right\|_p + \left\| f_- - s_m^- \right\|_p \to 0 \quad \text{as } m \to \infty.$$

Step 3. $f \in L^p(\mathbb{R}^n)$.

Given $\varepsilon > 0$, using Proposition 4.14, take $g \in C_c(\mathbb{R}^n)$ such that $\|f - g\|_p < \dfrac{\varepsilon}{2}$. By Step 2, take s_M such that $\|g - s_M\|_p < \dfrac{\varepsilon}{2}$. Hence,

$$\|f - s_M\|_p \leq \|f - g\|_p + \|g - s_M\|_p < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

(12) Let X be a metric space consisting of infinitely many elements and μ a Borel measure on X such that $\mu(B) > 0$ on any metric ball (i.e. $B = \{x : d(x, x_0) < \rho\}$ for some $x_0 \in X$ and $\rho > 0$. Show that $L^\infty(\mu)$ is non-separable.

Suggestion: Find disjoint balls $B_{r_j}(x_j)$ and consider $\chi_{B_{r_j}(x_j)}$.

Solution: We assume the existence of the sequence of disjoint balls $B_{r_j}(x_j)$

and prove the result. Obviously the subset of $L^\infty(\mu)$

$$A = \Big\{ \sum_{i=1}^{\infty} a_i \chi_{B_{r_j}(x_j)}, a_i = 0, 1 \Big\} \text{ is uncountable,}$$

let D be any dense set in $L^\infty(\mu)$, fix $a \in A$, $\exists y_a \in D$ s.t

$$d(y_a, a) < \frac{1}{3} \text{ and } y_a \neq y_b \text{ if } a \neq b.$$

Result follows from the uncountability of $\{y_a, a \in A\}$. It remains to prove the existence of disjoints balls. We claim that if there is a countable subset $J = \{x_i\}$ such that $\forall j$, x_j is not a limit point of J, then there is sequence of disjoint balls. $\exists r_1 > 0$, such that $B_{2r_1}(x_1) \cap J \setminus \{x_1\} = \emptyset$. Let $\overline{B_r(y)}$ be closure of the ball $B_r(y)$, $\exists r_2 > 0$, such that $B_{2r_2}(x_2) \subseteq \overline{B_{r_1}(x_1)}^c$ and $B_{2r_2}(x_2) \cap J \setminus \{x_2\} = \emptyset$. We obtain the desired sequence of ball by repeating the process. Now if there are a point y and a countable F s.t y is the only limit point of F, then let $F \setminus \{y\}$ be our J. Otherwise, we can take any countable subset of the space be J.

(13) Show that $L^1(\mu)' = L^\infty(\mu)$ provided $(X, \mathfrak{M}, \mu)$ is σ-finite, i.e., $\exists X_j$, $\mu(X_j) < \infty$, such that $X = \bigcup X_j$.
Hint: First assume $\mu(X) < \infty$. Show that $\exists g \in L^q(\mu)$, $\forall q > 1$, such that

$$\Lambda f = \int f g \, d\mu, \quad \forall f \in L^p, \; p > 1.$$

Next show that $g \in L^\infty(\mu)$ by proving the set $\{x : |g(x)| \geq M + \varepsilon\}$ has measure zero $\forall \varepsilon > 0$. Here $M = \|\Lambda\|$.

Solution:

Step 1. $\mu(X) < \infty$.

In this case, Hölder's inequality implies that a continuous linear functional Λ on $L^1(X)$ has a restriction to $L^p(X)$ which is again continuous since

$$|\Lambda f| \leq \|\Lambda\| \, \|f\|_1 \leq \|\Lambda\| \, \mu(X)^{1/q} \, \|f\|_p \tag{2}$$

for all $p \geq 1$. By the proof for $p > 1$ in the lecture notes, we have the existence of a unique $v_p \in L^q(X)$ such that $\Lambda f = \int v_p f \, d\mu$ for all $f \in L^p(X)$. Moreover, since $L^r(X) \subset L^p(X)$ for $r \geq p$ (by Hölder's inequality) the uniqueness of v_p implies that v_p is, in fact, independent of p, i.e. this function (which we now call v) is in every $L^r(X)$-space for $1 < r < \infty$.

If we now pick some conjugate exponents q and p with $p > 1$ and choose $f = |v|^{q-2}\overline{v}$ in (2), we obtain

$$\begin{aligned}
\int |v|^q \, d\mu &= \Lambda f \\
&\leq \|\Lambda\| \, \mu(X)^{1/q} \left(\int |v|^{(q-1)p} \, d\mu \right)^{1/p} \\
&= \|\Lambda\| \, \mu(X)^{1/q} \, \|v\|_q^{q-1},
\end{aligned}$$

and hence $\|v\|_q \leq \|\Lambda\| \, \mu(X)^{1/q}$ for all $q < \infty$. We claim that $v \in L^\infty(X)$; in fact $\|v\|_\infty \leq \|\Lambda\|$. Suppose that $\mu(\{x \in X : |v(x)| > \|\Lambda\| + \varepsilon\}) = M > 0$. Then $\|v\|_q \geq (\|\Lambda\| + \varepsilon) M^{1/q}$, which exceeds $\|\Lambda\| \, \mu(X)^{1/q}$ if q is big enough. Thus $v \in L^\infty(X)$ and $\Lambda f = \int v f \, d\mu$ for all $f \in L^p(X)$ for any $p > 1$. If $f \in L^1(X)$ is given, then $\int |v||f| \, d\mu < \infty$. Replacing f by $f^k = f \chi_{\{x : |f(x)| \leq k\}}$, we note that $|f^k| \leq |f|$ and $f^k(x) \to f(x)$ pointwise as $k \to \infty$; hence, by dominated convergence, $f^k \to f$ in $L^1(X)$ and $v f^k \to v f$ in $L^1(X)$. Thus

$$\Lambda f = \lim_{k \to \infty} \Lambda f^k = \lim_{k \to \infty} \int v f^k \, d\mu = \int v f \, d\mu.$$

Step 2. $\mu(X) = \infty$.

The previous conclusion can be extended to the case that $\mu(X) = \infty$ but X is σ-finite. Then

$$X = \bigcup_{j=1}^{\infty} X_j$$

with $\mu(X_j)$ finite and with $X_j \cap X_k$ empty whenever $j \neq k$. Any $L^1(X)$ function f can be written as

$$f(x) = \sum_{j=1}^{\infty} f_j(x)$$

where $f_j = \chi_j f$ and χ_j is the characteristic function of X_j. $f_j \mapsto \Lambda f_j$ is then an element of $L^1(X_j)'$, and hence there is a function $v_j \in L^\infty(X_j)$ such that $\Lambda f_j = \int_{X_j} v_j f_j \, d\mu = \int_{X_j} v_j f \, d\mu$. The important point is that each v_j is bounded in $L^\infty(X_j)$ by the *same* $\|\Lambda\|$. Moreover, the function v, defined on all of X by $v(x) = v_j(x)$ for $x \in X_j$, is clearly measurable and bounded by $\|\Lambda\|$. Thus, we have $\Lambda f = \int_X vf \, d\mu$ by the countable additivity of the measure μ.
If there exist $v, w \in L^\infty(X)$ such that

$$\Lambda f = \int_X vf \, d\mu = \int_X wf \, d\mu, \quad \forall f \in L^1(X),$$

then

$$\int_X (v - w) f \, d\mu = 0, \quad \forall f \in L^1(X).$$

Suppose, on the contrary, that $(v - w) > 0$ on some $A \subset \mathfrak{M}$ with $0 < \mu(A) < \infty$. By taking $f = \chi_A$ one arrives at a contradiction. Thus, given $\Lambda \in L^1(X)$ there corresponds a unique $v \in L^\infty(X)$.

(14) (a) For $1 \leq p < \infty$, $\|f\|_p, \|g\|_p \leq R$, prove that

$$\int \left| |f|^p - |g|^p \right| d\mu \leq 2pR^{p-1} \|f - g\|_p .$$

(b) Deduce that the map $f \mapsto |f|^p$ from $L^p(\mu)$ to $L^1(\mu)$ is continuous.

Hint: Try $|x^p - y^p| \leq p|x - y|(x^{p-1} + y^{p-1})$.

Solution:

(a) Notice that $|x^p - y^p| \leq p|x - y|(x^{p-1} + y^{p-1})$, which follows form the mean value theorem applying to $h(x) = x^p$. Then it follows easily from Hölder's inequality that

$$\int ||f|^p - |g|^p| \, d\mu \leq 2pR^{p-1} \|f - g\|_p .$$

(b) This is a direct consequence of (a).