Sample Solutions of Assignment 2 for MATH3270B:2.5-2.6,3.1-3.2

Note: Any problems to the sample solutions, email Ms. Zhang Rong(rzhang@math.cuhk.edu.hk) directly.

Section 2.5:2,4,6,8,12

2.**Answer:**

Since $f(y) = ay + by^2$, the critical points are zeros of $f(y)$.
Then we get $y = 0$ or $y = -\frac{a}{b}$. From the graph, we can see that
$y = 0$ is asymptotically unstable; $y = -\frac{a}{b}$ is asymptotically stable.

4.**Answer:**

Since $f(y) = e^y - 1$, the critical points are zeros of $f(y)$.
Then we get $y = 0$ From the graph, we can see that
$y = 0$ is asymptotically unstable.

6.**Answer:**

Since $f(y) = -4(\arctan y)/(1 + y^2)$, the critical points are zeros of $f(y)$.
Then we get $y = 0$ From the graph, we can see that
$y = 0$ is asymptotically stable;

8.**Answer:**

Since $f(y) = -k(y - 1)^2$,the critical points are zeros of $f(y)$.
Then we get $y = 1$. From the graph,we can see that $y = 1$ is semistable.

12.**Answer:**

Since $f(y) = y^2(4 - y^2)$, the critical points are zeros of $f(y)$.
Then we get $y = 0$ or $y = \pm 2$. From the graph, we can see that
$y = 0$, asymptotically semi-stable;
$y = 2$, asymptotically stable;
$y = -2$, asymptotically unstable.

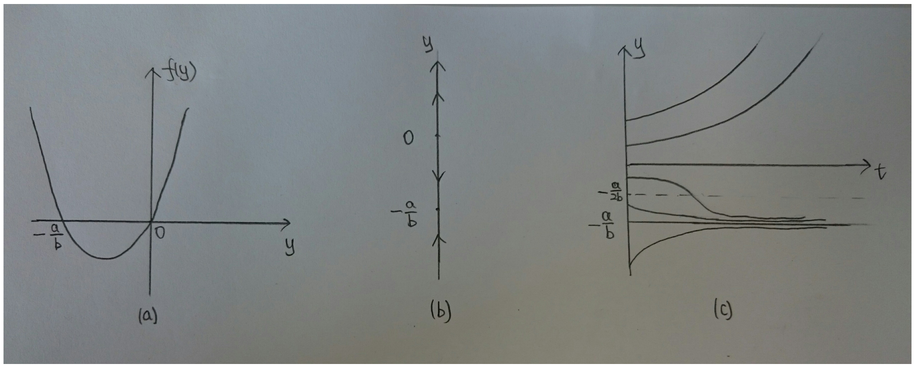

FIGURE 1. 2:(a)$f(y)$ versus y;(b)phase line;(c)graphs of solution

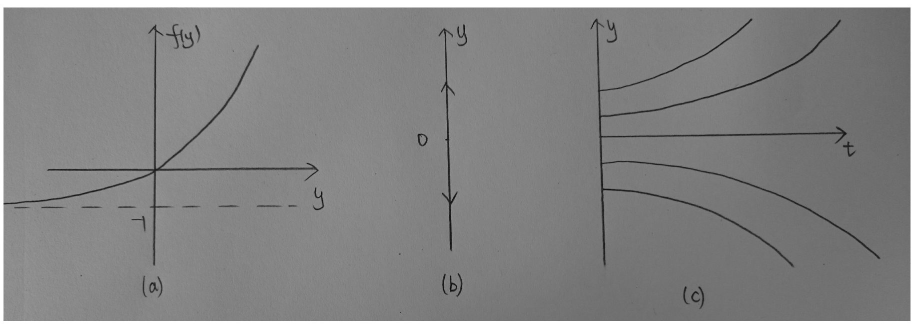

FIGURE 2. 4:(a)$f(y)$ versus y;(b)phase line;(c)graphs of solution

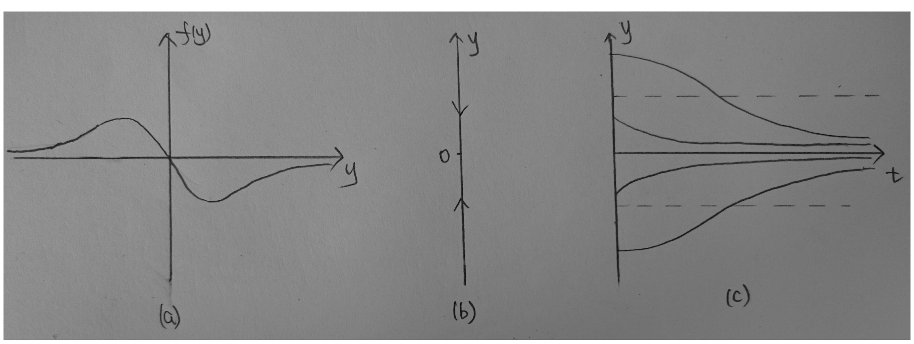

FIGURE 3. 6:(a)$f(y)$ versus y;(b)phase line;(c)graphs of solution

Section 2.6:1,2,4,8,10,11,18,20,22,25,27,32

1.**Answer:**

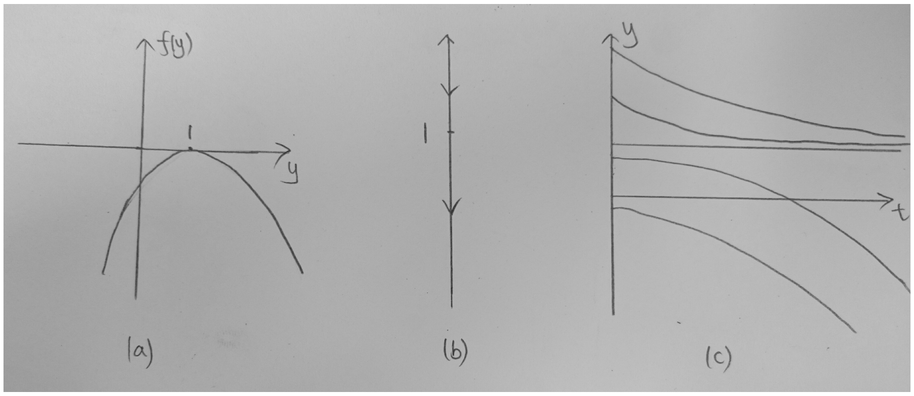

FIGURE 4. 8:(a)$f(y)$ versus y;(b)phase line;(c)graphs of solution

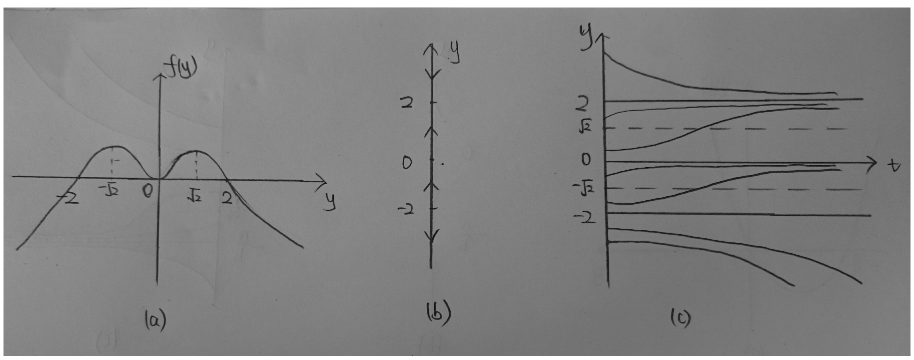

FIGURE 5. 12:(a)$f(y)$ versus y;(b)phase line;(c)graphs of solution

$M_y = 0 = N_x$,hence it's exact;

$$d(x^2 + 3x + y^2 - 2y) = 0$$
$$x^2 + 3x + y^2 - 2y = C$$

2.**Answer:**

$M_y = 4, N_x = 2$,$M_y \neq N_x$,hence it's not exact.

4.**Answer:**

$M_y = 4xy + 2 = N_x$ hence it's exact.

$$\begin{aligned} d(x^2y^2 + 2xy) &= 0 \\ x^2y^2 + 2xy &= C \end{aligned}$$

8.**Answer:**

$M_y = e^x \cos y + 3$, $N_x = -3 + e^x \sin y$,$M_y \neq N_x$,hence it's not exact.

10.**Answer:**

$M_y = \frac{1}{x}$, $N_x = \frac{1}{x}$,$M_y = N_x$,hence it's exact.

$$d(y \ln x + 3x^2 - 2y) = 0$$

$$y \ln x + 3x^2 - 2y = C$$

11.**Answer:**

$M_y = \frac{x}{y} + x$, $N_x = \frac{y}{x} + y$,$M_y \neq N_x$,hence it's not exact.

18.**Answer:**

Since $M_y = 0 = N_x$,hence it's exact;

20.**Answer:**

$M_y = \frac{\cos y}{y} - \frac{\sin y}{y^2}$,$N_x = \frac{-2e^{-x}\cos x - 2e^{-x}\sin x}{y}$,$M_y \neq N_x$,hence it's not exact.

$(M\mu)_y = \cos y e^x - 2\sin x = (N\mu)_x$,hence it's exact now.

$$\begin{aligned}(\sin y e^x - 2y \sin x)dx + (\cos y e^x + 2\cos x)dy &= 0\\ d(\sin y e^x + 2y\cos x) &= 0\\ \sin y e^x + 2y \cos x &= C\end{aligned}$$

22.**Answer:**

$M_y = (x+2)\cos y$,$N_x = \cos y$,$M_y \neq N_x$ so it's not exact.

$(M\mu)_y = xe^x(x+2)\cos y = (N\mu)_x$,hence it's exact now.

$$\begin{aligned}(xe^x(x+2)\sin y)dx + (x^2e^x \cos y)dy &= 0\\ d(\sin y x^2 e^x) &= 0\\ x^2 e^x \sin y &= C\end{aligned}$$

25.**Answer:**

Since $M_y = 3x^2 + 2x + 3y^2$, $N_x = 2x$,and $\frac{M_y - N_x}{N} = 3$, then there is an integrating factor that depends only on x, suppose $\mu(x,y) = \mu(x)$, and it satisfies $\mu_x = \dfrac{M_y - N_x}{N}\mu = 3\mu$. Hence

$$\mu = e^{3x}$$

Multiply μ on both sides of the equation,

$$(3x^2y + 2xy + y^3)e^{3x} + (x^2 + y^2)e^{3x}y' = 0$$

$$d(e^{3x}x^2y + \frac{1}{3}e^{3x}y^3) = 0$$

Hence the solutions are given by

$$e^{3x}x^2y + \frac{1}{3}e^{3x}y^3 = C$$

27.**Answer:**

Since $M_y = 0, N_x = \dfrac{1}{y}$,and $\dfrac{N_x - M_y}{M} = \dfrac{1}{y}$, then there is an integrating factor that depends only on y, suppose $\mu(x,y) = \mu(y)$, and it satisfies $\mu_y = \dfrac{N_x - M_y}{N}\mu = \dfrac{1}{y}\mu$. Hence

$$\mu = y$$

Multiply μ on both sides of the equation,

$$y + (x - \sin y y)y' = 0$$

$$d(xy + y\cos y - \sin y) = 0$$

Hence the solutions are given by

$$xy + y\cos y - \sin y = C$$

32.**Answer:**

Multiplying equation by μ,

$$\mu M = \frac{3x + y}{x(2x + y)}, \mu N = \frac{x + y}{y(2x + y)}$$

Since $(\mu M)_y = (\mu N)_x$, so it is exact now. Thus there is a $\psi(x,y)$ such that

$$\psi_x = \mu M$$

$$\psi_y = \mu N$$

Integrating second equation,we obtain

$$\psi = \frac{1}{2}\ln|y^2 + 2xy| + h(x)$$

By the first equation,we have

$$\psi_x = \frac{1}{y+2x} + h^{'}(x) = \mu N$$

Thus $h'(x) = \dfrac{1}{x}$ and $h(x) = \ln|x|$,then we have

$$\psi = \frac{1}{2}\ln|y^2 + 2xy| + \ln|x| = \frac{1}{2}\ln x^2|y^2 + 2xy|$$

Hence the solutions are given by

$$x^2(y^2 + 2xy) = C$$

Section 3.1:1,4,6,8,10,12,15,17,20,22

Find the general solution of the given differential equation

(1). $y'' + 2y' - 3y = 0$

(4). $2y'' - 3y' + y = 0$

(6). $4y^{''} - 9y = 0$

(8). $y^{''} - 2y^{'} - 2y = 0$

Answer:

(2)The characteristic equation is :

$$r^2 + 2r - 3 = 0$$

Thus $r_1 = -3$,$r_2 = 1$.

The general solution is $y = c_1e^{-3t} + c_2e^t$.

(4)The characteristic equation is :

$$2r^2 - 3r + 1 = 0$$

Thus $r_1 = \dfrac{1}{2}$,$r_2 = 1$.

The general solution is $y = c_1e^{\frac{1}{2}t} + c_2e^t$.

(6)The characteristic equation is :

$$4r^2 - 9 = 0$$

Thus $r_1 = \dfrac{3}{2}$,$r_2 = -\dfrac{3}{2}$.
The general solution is $y = c_1 e^{-\frac{3}{2}t} + c_2 e^{\frac{3}{2}t}$.
(8)The characteristic equation is :

$$r^2 - 2r - 2 = 0$$

Thus $r_1 = 1 + \sqrt{3}$,$r_2 = 1 - \sqrt{3}$.
The general solution is $y = c_1 e^{(1+\sqrt{3})t} + c_2 e^{(1-\sqrt{3})t}$.

(10). The characteristic equation is

$$r^2 + 4r + 3 = 0$$

Thus the possible values of r are $r = -1$ and $r = -3$, and the general solution of the equation is

$$y(t) = c_1 e^{-t} + c_2 e^{-3t}.$$

From the initial value, we have

$$c_1 + c_2 = 2$$

$$-c_1 - 3c_2 = -1,$$

hence $c_1 = \dfrac{5}{2}$ and $c_2 = -\dfrac{1}{2}$. The solution of the equation is

$$y(t) = \frac{5}{2} e^{-t} - \frac{1}{2} e^{-3t}.$$

$y(t) \longrightarrow 0$, as $t \longrightarrow \infty$.

(12).The characteristic equation is :

$$r^2 + 3r = 0$$

The roots are $r = 0, -3$.Hence the general solution is

$$y = c_1 + c_2 e^{-3t}$$

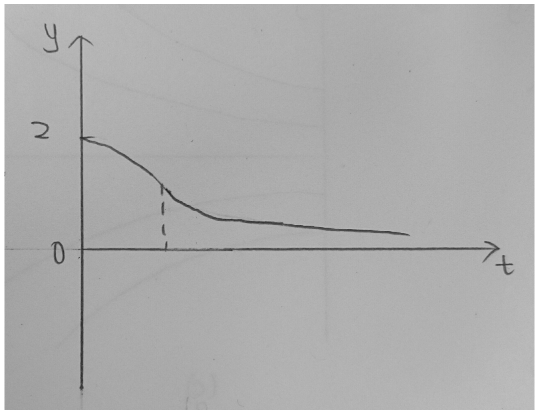

FIGURE 6. Question 10:graph of solution

According to the initial condition,the specific solution is

$$y = -1 - e^{-3t}$$

$y(t) \longrightarrow -1$, as $t \longrightarrow \infty$.

(15). The characteristic equation is

$$r^2 + 8r - 9 = 0$$

The roots are $r = -9, 1$. Hence the general solution is

$$y = c_1 e^{-9t} + c_2 e^t$$

Accounting for the initial conditions, the specific solution is

$$y(t) = \frac{1}{10} e^{-9(t-1)} + \frac{9}{10} e^{t-1}$$

$y(t) \longrightarrow \infty$, as $t \longrightarrow \infty$.

17.Find a differential equation whose general solution is $y = c_1 e^{2t} + c_2 e^{-3t}$

Answer: The the characteristic equation is

$$(r-2)(r+3) = r^2 + r - 6 = 0$$

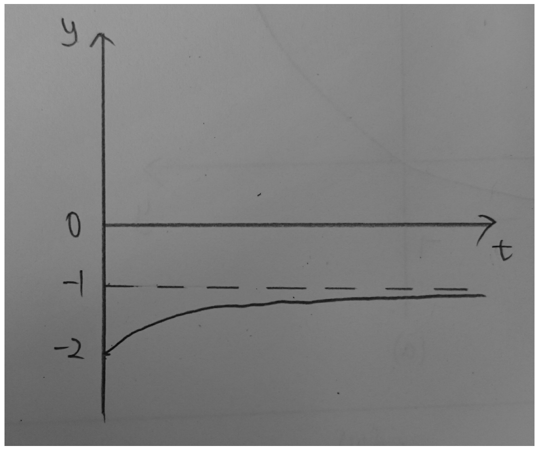

FIGURE 7. Question 12:graph of solution

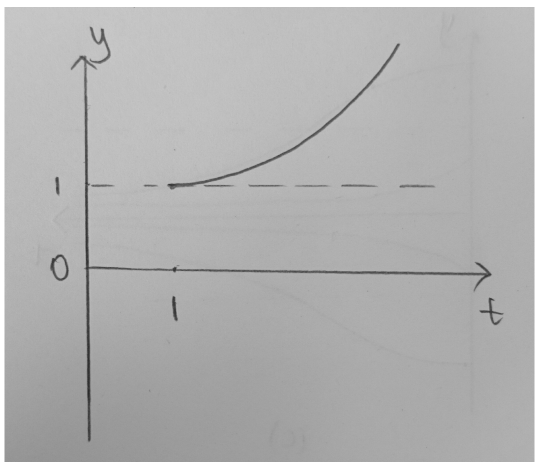

FIGURE 8. Question 15:graph of solution

So the equation is

$$y^{''} + y^{'} - 6y = 0.$$

20. Find the solution of the initial value problem

$$2y'' - 3y' + y = 0, \ y(0) = 2, \ y'(0) = \frac{1}{2}$$

Then determine the maximum value of the solution and also find the point where the solution is zero.

Answer: The characteristic equation is

$$2r^2 - 3r + 1 = (2r-1)(r-1) = 0$$

Thus the possible values of r are $r_1 = \frac{1}{2}$ and $r_2 = 1$, and the general solution of the equation is

$$y(t) = c_1 e^{\frac{1}{2}t} + c_2 e^t.$$

Using the first initial condition, we obtain

$$c_1 + c_2 = 2.$$

Using the second initial condition, we obtain

$$\frac{1}{2}c_1 + c_2 = \frac{1}{2}.$$

Solving above equations we find that $c_1 = 3$ and $c_2 = -1$. Hence,

$$y(t) = 3e^{\frac{1}{2}t} - e^t.$$

Since

$$y(t) = 3e^{\frac{1}{2}t} - e^t = \frac{9}{4} - (e^{\frac{1}{2}t} - \frac{3}{2})^2 \leq \frac{9}{4},$$

when $t = 2\ln\frac{3}{2}$, $y(t)$ reach the maximum value $\frac{9}{4}$. Solving $y(t) = 3e^{\frac{1}{2}t} - e^t = 0$, we know that the zero point of solution is $t = 2\ln 3$.

22. Solve the initial value problem $4y'' - y = 0,\ y(0) = 2,\ y'(0) = \beta$. Then find β so that the solution approaches zero as $t \to \infty$.

Answer: The the characteristic equation is

$$4r^2 - 1 = (2r+1)(2r-1) = 0$$

Thus the possible values of r are $r_1 = \frac{1}{2}$ and $r_2 = \frac{-1}{2}$, and the general solution of the equation is

$$y(t) = c_1 e^{\frac{1}{2}t} + c_2 e^{-\frac{1}{2}t}.$$

Using the first initial condition, we obtain

$$c_1 + c_2 = 2.$$

Using the second initial condition, we obtain

$$c_1 - c_2 = 2\beta.$$

By solving above equations we find that $c_1 = \beta + 1$ and $c_2 = 1 - \beta$. Hence,

$$y(t) = (\beta + 1)e^{\frac{1}{2}t} + (1 - \beta)e^{-\frac{1}{2}t}.$$

From $y(t) \to 0$ as $t \to \infty$, we find $\beta = -1$.

Section 3.2:2,4,5,7,9,12,15,16,17,20,23,25,28,31,33,38,39

Find the Wronskian of the given pair of functions.

(2). $\cos t,\ \sin t$

(4). x, xe^x

(5). $e^t \sin t,\ e^t \cos t$

Answer: The computation is easy, so we just give the final result.

(2). $W = 1$

(4). $W = x^2 e^x$

(5). $W = -e^{2t}$

In the following problems determine the longest interval in which the given initial value problem is certain to have a unique twice differentiable solution. Do not attempt to find the solution.

7.$ty'' + 3y = t, \quad y(1) = 1, \quad y'(1) = 2.$

9. $t(t-4)y'' + 3ty' + 4y = 2,\ y(3) = 0,\ y'(3) = -1$

12. $(x-2)y'' + y' + (x-2)(\tan x)y = 0, y(3) = 1, y'(3) = 2$

Answer: (7). The original solution can written as

$$y'' + \frac{3}{t}y = 1,$$

hence the longest interval is $t > 0$. Then the points of discontinuity of the coefficients are $t = 0$. Therefore, the longest open interval, containing the initial point $t = 1$, in which all the coefficients are continuous, is $0 < t < \infty$.

(9).The original solution can written as

$$y'' + \frac{3t}{t(t-4)}y' + \frac{4}{t(t-4)}y = \frac{2}{t(t-4)}$$

and $p(t) = \dfrac{3t}{t(t-4)}$, $q(t) = \dfrac{4}{t(t-4)}$, $g(t) = \dfrac{2}{t(t-4)}$. Then the points of discontinuity of the coefficients are $t = 0$ and $t = 4$. Therefore, the longest open interval, containing the initial point $t = 3$, in which all the coefficients are continuous, is $0 < t < 4$.

(12). The original solution can written as

$$y'' + \frac{y'}{x-2} + (\tan x)y = 0$$

Then the only points of discontinuity of the coefficients are $x = 2$, and $x = k\pi + \dfrac{1}{2}\pi, k \in \mathbb{Z}$. Therefore, the longest open interval, containing the initial point $x = 3$, in which all the coefficients are continuous, is $2 < x < \dfrac{3}{2}\pi$.

15. Show that if $y = \phi(t)$ is a solution of the differential equation $y^{''} + p(t)y^{'} + q(t)y = g(t)$, where $g(t)$ is not always zero, the $y = c\phi(t)$, where c is any constant other than 1, is not a solution. Explain why this result does not contradict the remark following Theorem 3.2.2.

Answer:

$$[c\phi(t)]^{''} + p(t)[c\phi(t)]^{'} + q(t)[c\phi(t)]$$

$$= c[\phi(t)^{''} + p(t)\phi(t)^{'} + q(t)\phi(t)]$$

$$= cg(t) \neq g(t)$$

if c is a constant other than 1, and $g(t)$ is not always zero.

This result does not contradict Theorem 3.2.2 because this equation is not homogeneous.

16. Can $y = \sin t^2$ be a solution on an interval containing $t = 0$ of an equation $y'' + p(t)y' + q(t)y = 0$ with continuous coefficients? Explain your answer.

Answer: By direct computing, we have

$$y = \sin t^2, \quad y' = 2t\cos t^2, \quad y'' = -4t^2 \sin t^2 + 2\cos t^2$$

we have

$$(-4t^2 + q(t))\sin t^2 + (2tp(t) + 2)\cos t^2 = 0 \qquad (*)$$

Assume that $y = \sin t^2$ is a solution on an interval containing $t = 0$ and coefficients are continuous, substitute $t = 0$ into (*), we get $2\cos 0 = 0$ which is not true. So $y = \sin t^2$ can not satisfy all the requirements.

17. If the Wronskian W of f and g is $3e^{4t}$, and if $f(t) = e^{2t}$, find $g(t)$.

Answer:

$$W = f(t)g^{'}(t) - f^{'}(t)g(t) = e^{2t}g^{'}(t) - 2e^{2t}g(t)$$

Let $W = 3e^{4t}$, we get the following equation

$$g^{'} - 2g(t) = 3e^{2t}.$$

From the above equation, $g(t) = 3te^{2t} + ce^{2t}$.

20. If the Wronskian of f and g is $t\cos t - \sin t$ and if $u = f + 3g,\ v = f - g$, find the Wronskian of u and v.

Answer:

$$\begin{aligned} W(u,v) &= uv^{'} - u^{'}v \\ &= (f+3g)(f^{'} - g^{'}) - (f^{'} + 3g^{'})(f - g) \\ &= -4fg^{'} + 4f^{'}g \\ &= -4W(f,g) \\ &= -4(t\cos t - \sin t). \end{aligned}$$

23. Find the fundamental set of solutions specified by Thm 3.2.5 for the given differential equation and initial point.

$$y'' + 4y' + 3y = 0, \quad t_0 = 1.$$

Answer: The characteristic equation is

$$r^2 + 4r + 3 = 0,$$

we have $r = -1$ and $r = -3$. The general solution is $y(t) = c_1 e^{-t} + c_2 e^{-3t}$.

If the initial condition $y(1) = 1, y'(1) = 0$ is satisfied ,then $y = -\frac{1}{2}e^{-3(t-1)} + \frac{3}{2}e^{-(t-1)}$

If the initial condition $y(1) = 0, y'(1) = 1$ is satisfied ,then $y = -\frac{1}{2}e^{-3(t-1)} + \frac{1}{2}e^{-(t-1)}$

So the set of fundamental solutions is $\{-\frac{1}{2}e^{-3(t-1)} + \frac{3}{2}e^{-(t-1)}, -\frac{1}{2}e^{-3(t-1)} + \frac{1}{2}e^{-(t-1)}\}$

25.**Answer:**

$$\begin{aligned} y_1'' - 2y_1' + y_1 &= e^t - 2e^t + e^t = 0 \\ y_2'' - 2y_2' + y_2 &= 2e^t + te^t - 2te^t - 2e^t + te^t = 0 \end{aligned}$$

Hence y_1 y_2 are the solutions of the ODE.

$$\begin{aligned} W(y_1, y_2) &= y_1 y_2' - y_1' y_2 \\ &= e^{2t} \neq 0 \end{aligned}$$

Hence they constitute the fundamental set of solutions.

28.**Answer:**

(a) Since the Wronskian $W = 3e^t$, they form a fundamental set of solutions.

(b) Since they are all linear combinations of y_1 and y_2, they are also solutions of the given equation.

(c) By calculating the respective Wronskians, we see that $[y_1, y_3]$ and $[y_1, y_4]$ are fundamental set of solutions while the others are not.

31.**Answer:**

Divide by x^2 on both sides: $y'' + \frac{1}{x}y' + (1 - \frac{v^2}{x^2})y = 0$

By Abel's theorem:

$$\begin{aligned} W(y_1, y_2) &= c\exp[-\int \frac{1}{x}dx] \\ &= c\exp[-\ln|x|] \\ &= \frac{c}{|x|} \end{aligned}$$

34.**Answer:**

$$y'' + \frac{2}{t}y' + e^t y = 0$$

$$\begin{aligned} W(y_1, y_2)(t) &= c\exp[-\int \frac{2}{t}dt] \\ &= c\exp[-2\ln|t|] \\ &= \frac{c}{t^2} \end{aligned}$$

Since $W(y_1, y_2)(1) = c = 2,\ W(y_1, y_2)(5) = \frac{2}{25}$.

38.**Answer:**

Since $W(y_1, y_2)(t) = y_1y_2' - y_1'y_2$,suppose that $y_1(t_0) = y_2(t_0) = 0$,then $W(y_1, y_2)(t_0) = 0$.

By Abel's Theorem, $W(y_1, y_2)(t_0) = c\exp[-\int p(t)dt] = 0$,thus $c = 0$,hence $W \equiv 0$.

Thus they cannot constitute the fundamental set of solutions.

39.**Answer:**

If so then exist a point t_0 such that $y_1(t_0)' = y_2(t_0)' = 0$, hence $W(y_1, y_2)(t_0) = 0$.

By Abel's Theorem,$W(y_1, y_2)(t_0) = c\exp[-\int p(t)dt] = 0$,thus $c = 0$,hence $W \equiv 0$.

Thus they cannot constitute the fundamental set of solutions.