Sample Solutions of Assignment 1 for MATH3270A: 1.1-1.3, 2.1-2.3

Note: Any problems about the sample solutions, please email Mr.YUAN, Yuan (yyuan@ math.cuhk.edu.hk) directly.

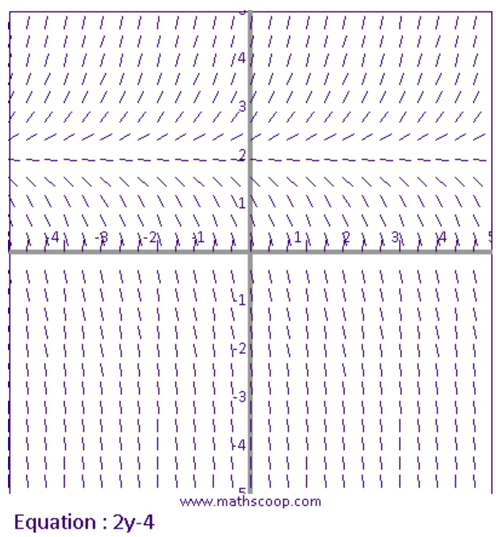
May,2015

1.1

Draw a direction field for the given direential equation. Based on the direction field, determine the behavior of y as $t \to \infty$.

2. $y' = 2y - 4$.

Solution: The direction field is shown in the following. According to the direction



field, we have

(a) If $y(0) > 2$, then $y' > 0$ and as y increases, y' increases. So as $t \to \infty$, $y \to +\infty$.

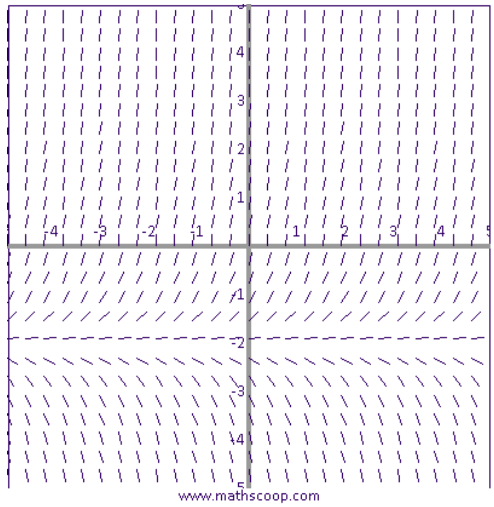
(b) If $y(0) = 0$, then $y' = 0$. Therefore $y = 2$ is the equilibrium solution. So as $t \to \infty$, $y \to 2$.

(c) If $y(0) < 2$, then $y' < 0$ and as y decreases, y' decreases. So as $t \to \infty$, $y \to -\infty$.

3. $y' = 4 + 2y$.

Solution: The direction field is shown in the following.

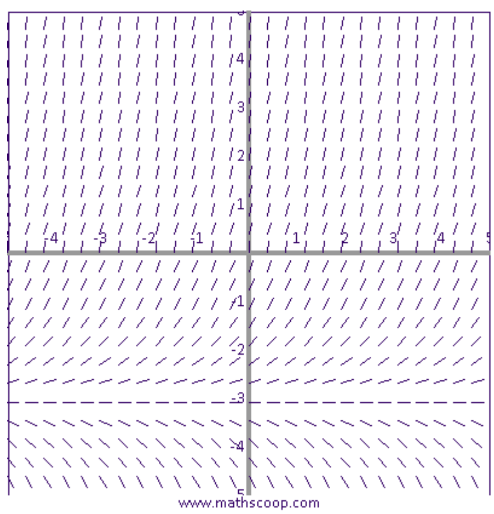
The analysis is similar to Problem 2.



Equation : 4+2y

6. $y' = y + 3$.

   Solution: The direction field is shown in the following.



Equation : y+3

   The analysis is similar to Problem 2.

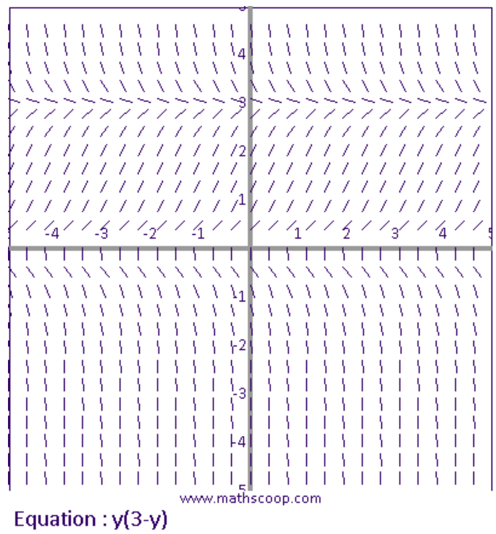
11. $y' = y(3 - y)$.

    Solution: The direction field is shown in the following.

    (a) If $y(0) > 3$, then $y' < 0$ and as y decreases, y' increases. So as $t \to \infty$, $y \to 3$.

    (b) If $y(0) = 0$ or $y = 3$, then $y' = 0$. Therefore $y = 0, 3$ are the equilibrium solutions.

    (c) If $0 < y(0) < 3$, then $y' > 0$ and as y increases, y' decreases. So as $t \to \infty$, $y \to 3$.



(d) If $y(0) < 0$, then $y' < 0$ and as y decreases, y' decreases. So as $t \to \infty$, $y \to -\infty$.

25. **Solution:**

(a) Assume the velocity of a falling object is $v(t)$. Then the differential equation is

$$\frac{dv}{dt} = g - \frac{A}{m}v^2,$$

where $g = 9.8$ is the acceleration of gravity, and A is drag coefficient.

(b) Let $\frac{dv}{dt} = 0$, then $v = \sqrt{\frac{mg}{A}}$. At the beginning of the falling procedure, $v = 0$ and $\frac{dv}{dt} > 0$. As v increases, $\frac{dv}{dt}$ decreases. So the limit velocity is

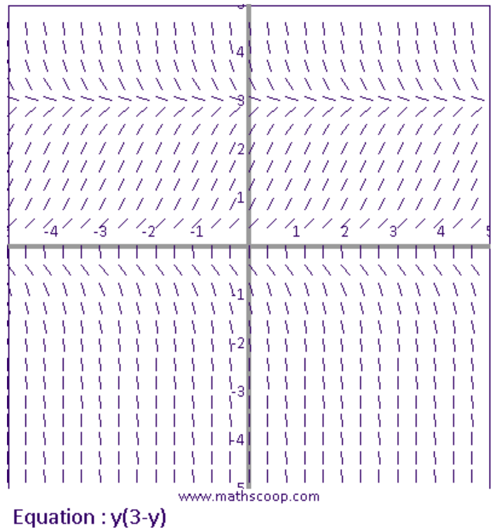
$$v = \sqrt{\frac{mg}{A}}$$

(c) Use the above equation, $A = mg/v^2 = \frac{2}{49}$.

(d) $\frac{dv}{dt} = \frac{49}{5} - \frac{v^2}{245}$.

1.2

1 Solve the following initial value problems and plot the solutions of several value of y_0. Then describe in a few words how the solutions resemble, and differ from each other.



(a) $dy/dt = -y + 3$, $y(0) = y_0$.

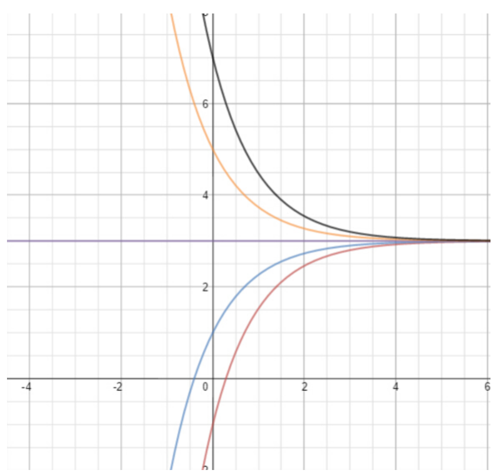
Solution: Note that $y = 3$ is one solution for the differential equation. If $y \neq 3$, then we have

$$\begin{aligned} \frac{dy}{-y+3} &= dt, \\ \ln|y-3| &= -t + C, \\ y &= 3 \pm e^{-t+C}, \end{aligned}$$

where C is a constant. The initial value $y(0) = y_0$ gives

$$y(t) = 3 + (y_0 - 3)e^{-t}.$$

Select the following initial value $y_0 = -1, 1, 3, 5, 7$. Any solution of $y_0 < 3$



satisfies that if t increases, y increases and y' decreases, and $y \to 3$ as $t \to \infty$.

Any solution of $y_0 > 3$ satisfies that if t increases, y decreases and y' increases, and $y \to 3$ as $t \to \infty$. The solution $y = 3$ is keeping unchanged.

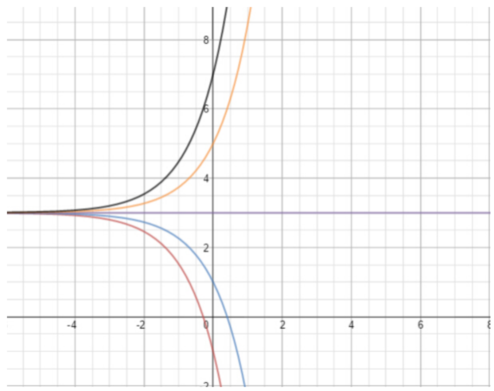
(b) The same as (a). Omit here.

(c) The same as (a). Omit here.

2(a). **Solution:** Omit the process. The solution is

$$y(t) = 3 + (y_0 - 3)e^t.$$

Any solution of $y_0 < 3$ satisfies that if t increases, y decreases and y' decreases, and



$y \to -\infty$ as $t \to \infty$. Any solution of $y_0 > 3$ satisfies that if t increases, y increases and y' increases, and $y \to +\infty$ as $t \to \infty$. The solution $y = 3$ is keeping unchanged.

3. **Solution:**

(a) $y = (b/a) + Ce^{-at}$.

(c) (i) Equilibrium is lower and is approached more rapidly. (ii) Equilibrium is higher. (iii)Equilibrium remains the same and is approached more rapidly.

7. **Solution:**

(a) By Problem 3, $p(t) = 900 + Ce^t/2$. $p(0) = 800$ gives $C = -100$. Let $p(T) = 0$, then we have $t = 4\ln 3 \approx 4.39$ (months).

(b) $T = 2(\ln \dfrac{900}{900 - p_0})$.

(c) Let $T \leq 12$, then $900 > p_0 \geq 900(1 - e^{-6}) \approx 897.8$.

1.3

1.

$$t^2 \frac{d^2y}{dt^2} + t\frac{dy}{dt} + 3y = \sin t$$

is of order 2 and nonlinear.

4.

$$\frac{dy}{dt} + ty^3 = 0$$

is of order 3 and linear.

6.

$$\frac{d^3y}{dt^3} + t\frac{dy}{dt}(\sin^2 t)y = t^3$$

is of order 3 and nonlinear.

9. Direct computation.

2.1

I did not any available software to draw the direction fields for such complex equations in the following. So I just solve the following equations. I omit the process in Problem 3, 5, 7.

1. $y' + 4y = t + e^{-2t}$.

   Solution: The integrating factor $\mu(t) = e^{4t}$. Multiplying the equation by the integrating factor gives

   $$(e^{4t}y)' = te^{4t} + e^{2t}.$$

   Then by integrating both sides of the above equation, we have

   $$e^{4t}y = \frac{1}{4}te^{4t} - \frac{1}{16}e^{4t} + \frac{1}{2}e^{2t} + C,$$

   where C is a constant. Therefore,

   $$y(t) = \frac{1}{4}t - \frac{1}{16} + \frac{1}{2}e^{-2t} + Ce^{-4t}.$$

3. $y' + y = te^{-t} + 2$.

   Solution: The integrating factor $\mu(t) = e^t$. The solution is

   $$y(t) = 2 + \frac{t^2e^{-t}}{2} + Ce^{-t}.$$

5. $y' - 3y = 4e^t$.

   Solution: The integrating factor $\mu(t) = e^{-3t}$. The solution is

   $$y(t) = -2e^t + Ce^{3t}.$$

7. $y' + 2ty = 2te^{-t^2}$.

   Solution: The integrating factor $\mu(t) = e^{t^2}$. The solution is

   $$y(t) = t^2e^{-t^2} + Ce^{-t^2}.$$

2.2

1. $y' = 3x^2/y$.

   Solution: Note that $y \neq 0$. Multiplying the above equation by y gives

   $$y\ dy = 3x^2\ dx.$$

   Integrate both sides of the above equation and we get

   $$\frac{y^2}{2} = x^3 + C,$$

   where C is an arbitrary constant.

   $$y = \pm\sqrt{2x^3 + 2C}$$

3. $y' + y^2 \cos x = 0$

   Solution: It is trivial to find $y = 0$ is one solution. If $y \neq 0$, ...(omit words)...,

   $$\begin{aligned} -\frac{dy}{y^2} &= \cos x\ dx \\ \frac{1}{y} &= \sin x + C \\ y &= \frac{1}{\sin x + C} \end{aligned}$$

   In conclusion, $y = 0$ and $y = \dfrac{1}{\sin x + C}$ (C is an arbitrary constant) are the solutions.

6. $xy' = (1 - y^2)^{1/2}$.

   Solution: Same as above, and omit the process.

   The solutions are

   $$y = \pm 1, \qquad y = \sin(\ln|x| + C). \tag{1}$$

8. $\dfrac{dy}{dx} = \dfrac{x^3}{2+y^2}$.
Solution: The solutions are

$$\frac{x^4}{4} - \frac{y^3}{3} - 2y + C = 0. \tag{2}$$

Remark: When solving the initial value problem, take care of the interval that the solution is valid (just like Problem 22).

2.3

3. Assume the amount of salt in the tank is $Q(t)$. The Q satisfies the following function in the first 10 minutes:

$$\frac{dQ}{dt} = 1 - \frac{Q}{50}, \qquad Q(0) = 0.$$

...(omit the process of solving separable equation)...Then $Q(t) = 50(1 - e^{-0.02t})$ for $0 \le t \le 10$. Q satisfies the following function in the later 10 minutes:

$$\frac{dQ}{dt} = -\frac{Q}{50}, \qquad Q(10) = 50(1 - e^{-0.2}).$$

Then $Q(t) = 50(1 - e^{-0.2})e^{0.2}e^{-0.02t}$ for $10 \le t \le 20$. So we have $Q(20) = 50(1 - e^{-0.2})e^{-0.2} \approx 7.42$ kg.