Tutorial 6.

Sect. 4.4 $\boxed{9}$ Find the sol of the given Initial value problem.

$$y''' + y' = \sec t \,; \quad y(0)=2,\ y'(0)=1,\ y''(0)=2.$$

Sol. The characteristic eqn for $y''' + y' = \sec t$ is

$$r^3 + r = 0$$

So the fundamental set of sols are

$$y_1(t) = 1, \quad y_2(t) = \sin t, \quad y_3(t) = \cos t.$$

Then the Wronskian $W(y_1, y_2, y_3) = \begin{vmatrix} 1 & \sin t & \cos t \\ 0 & \cos t & -\sin t \\ 0 & -\sin t & -\cos t \end{vmatrix} = -1$

$$W_1(t) = \begin{vmatrix} 0 & \sin t & \cos t \\ 0 & \cos t & -\sin t \\ 1 & -\sin t & -\cos t \end{vmatrix} = \begin{vmatrix} \sin t & \cos t \\ \cos t & -\sin t \end{vmatrix} = -1$$

$$W_2(t) = \begin{vmatrix} 1 & 0 & \cos t \\ 0 & 0 & -\sin t \\ 0 & 1 & -\cos t \end{vmatrix} = \sin t$$

$$W_3(t) = \begin{vmatrix} 1 & \sin t & 0 \\ 0 & \cos t & 0 \\ 0 & -\sin t & 1 \end{vmatrix} = \cos t$$

Therefore, a particular sol. for $y''' + y' = \sec t$ is

$$Y_0(t) = (1)\int_0^t \frac{\sec s\, W_1(s)}{W(s)}\,ds + (\sin t)\int_0^t \frac{\sec s\, W_2(s)}{W(s)}\,ds$$

$$+ (\cos t)\int_0^t \frac{\sec s\, W_3(s)}{W(s)}\,ds$$

$$= \int_0^t \sec s\,ds + \sin t\int_0^t -\tan s\,ds + \cos t\int_0^t 1\,ds$$

$$= \ln\left(\sec t + \tan t\right) + (\sin t)\ln\cos t + t\cos t.$$

$$\left(\text{when } t \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)\quad \sec t + \tan t > 0,\ \cos t > 0\right).$$

Then we conclude the general sol is

$$Y(t) = c_1 + c_2\sin t + c_3\cos t + \ln(\sec t + \tan t)$$
$$+ (\sin t)\ln\cos t + t\cos t.$$

The initial condition gives

$$c_1 = 0,\quad c_2 = 1,\quad c_3 = 2.$$

$\square$