Tutorial 4

3.2 $\boxed{40}$ Prove that: if y_1 & y_2 have a common point of interflection t_0 in I. then they cannot be a fundamental set of solutions on I unless both p & q are zero at t_0.

Proof: ~~Assume y_1 & y_2 hav~~

Assume p & q are not both zero at t_0.

then $y_1''(t_0) = y_2''(t_0) = 0$ and the eqn.

$$y'' + p(t)\, y' + q(t)\, y = 0$$

imply that

$$\begin{cases} p(t_0)\, y_1'(t_0) + q(t_0)\, y_1(t_0) = 0 \\ p(t_0)\, y_2'(t_0) + q(t_0)\, y_2(t_0) = 0 \end{cases}$$

Since $(p(t_0), q(t_0)) \neq 0$. $\begin{vmatrix} y_1'(t_0) & y_1(t_0) \\ y_2'(t_0) & y_2(t_0) \end{vmatrix} = 0$.

Then $W(y_1, y_2)(t_0) = \begin{vmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{vmatrix} = 0$.

By Abel's theorem, $W(y_1, y_2)(t) = W(y_1, y_2)(t_0) \exp\left(-\int_{t_0}^{t} p(s)\,ds\right)$

$= 0$.

So ——.

$\square$

3.4 $\boxed{21}$ Use the method of reduction of order to find a second solution of the given differential equation.

$$t^2y''+ty'-4y=0, \quad t>0; \quad y_1(t)=t^2$$

Sol. Let $y=v(t)\,y_1(t)=t^2v(t)$

then the equation implies

$$t^2(2v+4tv'+t^2v'')+t(2tv+t^2v')-4t^2v=0$$

$$t^4v''+5t^3v'=0$$

$$tv''+5v'=0$$

$$\Rightarrow v'=t^{-5}+C$$

$$\Rightarrow y=v(t)\,t^2=-\frac{t^{-2}}{4}+ct^3$$

Choosing $c=0$ gives a new solution $y_2=-\frac{t^{-2}}{4}$.

The Wronskian of y_1 & y_2 is.

$$W(y_1,y_2)(t)=\begin{vmatrix} t^2 & -\frac{t^{-2}}{4} \\ 2t & \frac{t^{-3}}{2} \end{vmatrix}=t^{-1}\neq 0.$$

Therefore y_1 & y_2 consists of a fundament set of solution.

$\square$

3.6 $\boxed{17}$ Verify the given y_1 & y_2 satisfies the corresponding homogeneous equation; then find a particular solution of the given nonhomogeneous equation.

$x^2y'' - 3xy' + 4y = x^2\ln x, \; x>0; \quad y_1(x) = x^2, \; y_2(x) = x^2\ln x$

Sol. The corresponding homogeneous equation is

$$x^2y'' - 3xy' + 4y = 0.$$

(Verification ...)

$$\cancel{y''} - \frac{3}{x}y' + \frac{4}{x^2}y = \ln x. \quad \text{(standard form)}$$

Therefore a particular sol. of the nonhomogeneous eqn. is

$$Y(x) = -x^2\int_{x_0}^{x} \frac{s^2\ln s\,\ln s}{W(y_1,y_2)(s)}\,ds + x^2\ln x\int_{x_0}^{x}\frac{s^2\ln s}{W(y_1,y_2)(s)}\,ds$$

$$W(y_1,y_2)(s) = \begin{vmatrix} s^2 & s^2\ln s \\ 2s & 2s\ln s + s \end{vmatrix} = s^3$$

$$Y(x) = -x^2\int_{x_0}^{x}\frac{(\ln s)^2}{s}\,ds + x^2\ln x\int_{x_0}^{x}\frac{\ln s}{s}\,ds$$

$$= \frac{x^2(\ln x)^3}{6} + C_0$$

See the back page (...) $\Rightarrow C_0 = 0$. $\quad Y(t) = \frac{1}{6}x^2(\ln x)^3$ $\square$

$$y(x) = \frac{1}{6}x^2(\ln x)^3 + C_0$$

$$y'(x) = \frac{x}{3}(\ln x)^3 + \frac{x}{2}(\ln x)^2$$

$$y''(x) = \frac{(\ln x)^3}{3} + (\ln x)^2 + \frac{(\ln x)^2}{2} + \ln x$$

$$= \frac{(\ln x)^3}{3} + \frac{3}{2}(\ln x)^2 + \ln x$$

The equation $x^2y'' - 3xy' + 4y = x^2\ln x$

gives $C_0 = 0$.