Lecture 1
Introduction to Integer Programming

MATH3220 Operations Research and Logistics
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Agenda

1. **Intro to IP**
2. **Some IP Models**

Linear Programming

$$
\begin{aligned}
\text{max (or min)} \quad & z = c_1 x_1 + c_2 x_2 + \cdots + c_n x_n \\
\text{s.t.} \quad & a_{11} x_1 + a_{12} x_2 + \cdots + a_{1n} x_n \le b_1 \\
& a_{21} x_1 + a_{22} x_2 + \cdots + a_{2n} x_n \le b_2 \\
& \qquad \vdots \qquad\qquad\qquad\qquad\qquad\quad \vdots \\
& a_{m1} x_1 + a_{m2} x_2 + \cdots + a_{mn} x_n \le b_m \\
& x_j \ge 0 \text{ for each } j = 1, \ldots, n
\end{aligned}
$$

In general,

$$
\begin{aligned}
\text{max (or min)} \quad & z = cx \\
\text{s.t.} \quad & Ax \le b \\
& x \ge 0
\end{aligned}
$$

A 2-variable integer programming

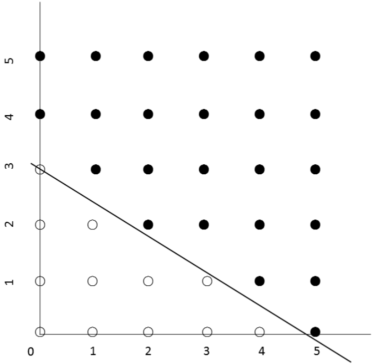
$$\begin{array}{ll} \text{maximize} & 3x + 4y \\ \text{subject to} & 5x + 8y \leq 24 \\ & x, y \geq 0 \text{ and integers} \end{array}$$

Q: What is the optimal solution?

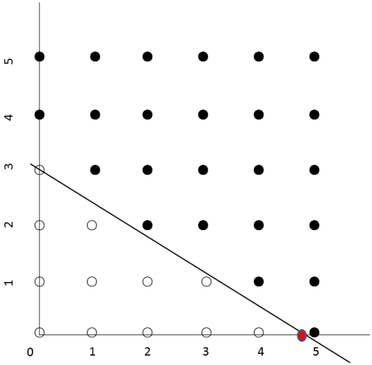
Feasible Region

Question 1: What is the optimal integer solution?
Question 2: What is the optimal linear solution?
Question 3: Can we use linear programming to solve this integer programming?

$$\begin{aligned} \text{maximize} \quad & 3x + 4y \\ \text{subject to} \quad & 5x + 8y \leq 24 \\ & x, y \geq 0 \text{ and integers} \end{aligned}$$

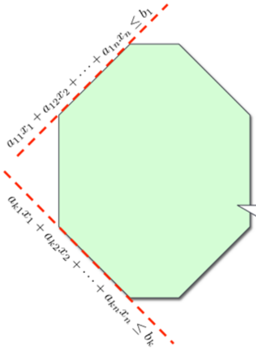


A rounding technique that sometimes is useful, and sometimes not.



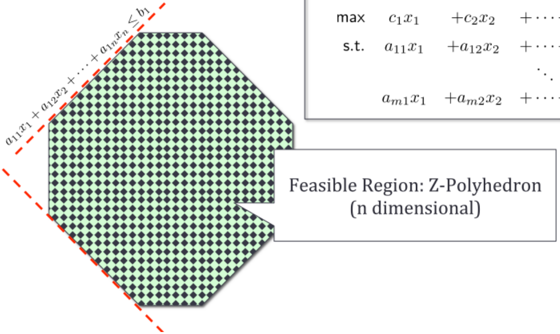
- Solve LP (ignore integrality), get $x^* = \frac{24}{5}, y^* = 0$ and $z^* = 14\frac{2}{5}$.
- Round, get $x = 5, y = 0$, infeasible!
- Truncate, get $x = 4, y = 0$, and $z = 12$.
- Same solution value at $x = 0, y = 3$.
- However, the optimal is $x = 1, y = 1$ and $z = 13$.

Linear vs. integer programming



$$
\begin{array}{lcccccc}
\text{max} & c_1x_1 & +c_2x_2 & +\cdots+ & c_nx_n & & \\
\text{s.t.} & a_{11}x_1 & +a_{12}x_2 & +\cdots+ & a_{1n}x_n & \leq & b_1 \\
 & & & \ddots & & \vdots & \\
 & a_{m1}x_1 & +a_{m2}x_2 & +\cdots+ & a_{mn}x_n & \leq & b_m
\end{array}
$$

Linear vs. integer programming



$$
\begin{array}{llllllll}
\text{max} & c_1x_1 & +c_2x_2 & +\cdots+ & c_nx_n & & \\
\text{s.t.} & a_{11}x_1 & +a_{12}x_2 & +\cdots+ & a_{1n}x_n & \leq & b_1 \\
 & & & \ddots & & \vdots & \\
 & a_{m1}x_1 & +a_{m2}x_2 & +\cdots+ & a_{mn}x_n & \leq & b_m
\end{array}
$$

Linear vs. integer programming

linear programming

$$
\begin{aligned}
\max \quad & c_1x_1 + c_2x_2 + \cdots + c_nx_n \\
\text{s.t.} \quad & a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \leq b_1 \\
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& \qquad \vdots \qquad\qquad\qquad\qquad\qquad \vdots \\
& a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \leq b_m \\
& x_j \geq 0 \text{ for each } j = 1, \ldots, n
\end{aligned}
$$

Linear vs. integer programming

integer programming (IP)

$$
\begin{array}{lll}
\max & c_1 x_1 + c_2 x_2 + \cdots + c_n x_n & \\
\text{s.t.} & a_{11} x_1 + a_{12} x_2 + \cdots + a_{1n} x_n & \leq b_1 \\
& a_{21} x_1 + a_{22} x_2 + \cdots + a_{2n} x_n & \leq b_2 \\
& \vdots & \vdots \\
& a_{m1} x_1 + a_{m2} x_2 + \cdots + a_{mn} x_n & \leq b_m \\
& x_j \geq 0 \quad \text{and integer} &
\end{array}
$$

Linear vs. integer programming

integer programming(binary variable)

$$
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\max \quad & c_1 x_1 + c_2 x_2 + \cdots + c_n x_n \\
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& a_{m1} x_1 + a_{m2} x_2 + \cdots + a_{mn} x_n \le b_m \\
& x_j = 0 \text{ or } 1
\end{aligned}
$$

Linear vs. integer programming

integer programming (binary variable)

$$
\begin{aligned}
\max \quad & c_1 x_1 + c_2 x_2 + \cdots + c_n x_n \\
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& \vdots \qquad\qquad\qquad\qquad \vdots \\
& a_{m1} x_1 + a_{m2} x_2 + \cdots + a_{mn} x_n \le b_m \\
& x_j = 0 \text{ or } 1 \Leftrightarrow 0 \le x_j \le 1, x_j \text{ integer}
\end{aligned}
$$

Types of integer programming

Integer programming problems usually involve optimization of a linear objective function to linear constraints, nonnegativity conditions and some or all of the variables are required to be integer.

A pure integer program: All variables are required to be integral.

$$\begin{aligned} \max \quad & 3x_1 + 4x_2 + 5x_3 + 6x_4 \\ \text{s.t.} \quad & 2x_1 + 3x_2 - 4x_3 + 2x_4 \leq 34 \\ & x_1 + x_2 + x_3 + x_4 \leq 9 \\ & x_j \geq 0 \text{ and integer for each } j = 1 \text{ to } 4 \end{aligned}$$

In general,

$$\begin{aligned} \text{max (or min)} \quad & z = cx \\ \text{s.t.} \quad & Ax = b, x \geq 0 \text{ integer} \end{aligned}$$

Types of integer programming

Binary (or 0-1) integer program: All variables are required to be 0 or 1.

$$\begin{aligned} \max \quad & 3x_1 + 4x_2 + 5x_3 + 6x_4 \\ \text{s.t.} \quad & 2x_1 + 3x_2 - 4x_3 + 2x_4 \leq 34 \\ & x_1 + x_2 + x_3 + x_4 \leq 9 \\ & x_j \in \{0, 1\} \text{ for each } j = 1 \text{ to } 4 \end{aligned}$$

Recall: the constraint

$$x_j \in \{0, 1\}$$

is equivalent to

$$0 \leq x_j \leq 1 \text{ and } x_j \text{ is integer}$$

Types of integer programming

Mixed integer program(MILP)

Some but not necessarily all variables are required to be integer. Other variables are permitted to be fractional.

$$\begin{aligned} \max \quad & 3x_1 + 4x_2 + 5x_3 + 6x_4 \\ \text{s.t.} \quad & 2x_1 + 3x_2 - 4x_3 + 2x_4 \leq 34 \\ & x_1 + x_2 + x_3 + x_4 \leq 9 \\ & x_j \in \{0, 1\} \text{ for each } j = 1, 2 \\ & x_3, x_4 \geq 0 \end{aligned}$$

In general,

$$\begin{aligned} \max \text{ (or min)} \quad & z = c_1 x + c_2 v \\ \text{s.t.} \quad & A_1 x + A_2 v = b \\ & x \geq 0 \text{ integer} \\ & v \geq 0. \end{aligned}$$

Some IP Models

Capital budgeting problem

investment budget = $14,000

Investment	1	2	3	4	5
Cash Required	$5,000	$4,000	$7,000	$3,000	$6,000
Present Value	$12,000	$11,000	$13,000	$8,000	$15,000

An investment can be selected or not. One cannot select a fraction of an investment.

Question: How to place the money so as to maximize the total present value?

Capital budgeting problem - con't

investment budget = \$14,000

Investment	1	2	3	4	5
Cash Required	\$5,000	\$4,000	\$7,000	\$3,000	\$6,000
Present Value	\$12,000	\$11,000	\$13,000	\$8,000	\$15,000

- Decision variables

$$x_j = \begin{cases} 1, & \text{if we invest in } j = 1, \ldots, 5 \\ 0, & \text{otherwise} \end{cases}$$

- Objective and constraints

$$\begin{array}{ll} \max & 12x_1 + 11x_2 + 13x_3 + 8x_4 + 15x_5 \\ \text{s.t.} & 5x_1 + 4x_2 + 7x_3 + 3x_4 + 6x_5 \leq 14 \\ & x_j \in \{0, 1\} \text{ for each } j = 1 \text{ to } 5 \end{array}$$

Capital budgeting problem - con't

How to model "logical" constraints

- Only one of the previous 4 investments can be accept.
- We can only make two investments.
- If investment 1 is made, investment 2 must also be made.
- If investment 1 is made, investment 3 cannot be made.
- Either investment 2 is made or investment 3 is made, but not both.

Formulating Constraints

- Only one of the previous 4 investments can be accept.

$$x_1 + x_2 + x_3 + x_4 \leq 1$$

Formulating Constraints

- Only one of the previous 4 investments can be accept.

$$x_1 + x_2 + x_3 + x_4 \leq 1$$

- We can only make two investments.

$$x_1 + x_2 + x_3 + x_4 + x_5 \leq 2$$

Formulating Constraints

- Only one of the previous 4 investments can be accept.

$$x_1 + x_2 + x_3 + x_4 \leq 1$$

- We can only make two investments.

$$x_1 + x_2 + x_3 + x_4 + x_5 \leq 2$$

- If investment 1 is made, investment 2 must also be made.

$$x_2 \geq x_1$$

Formulating Constraints

- If investment 1 is made, investment 3 cannot be made.

$$x_1 + x_3 \leq 1$$

Formulating Constraints

- If investment 1 is made, investment 3 cannot be made.

$$x_1 + x_3 \leq 1$$

- Either investment 2 is made or investment 3 is made, but not both.

$$x_2 + x_3 = 1$$

Capital budgeting problem - con't

In general, assume that there are n potential investments. In particular, investment j has a present value c_j, and requires an investment of a_{ij} amount of resource i, such as cash or manpower, used on the jth investment, we can state the problem formally as:

$$
\begin{array}{lll}
\text{Maximize} & \sum_{j=1}^{n} c_j x_j, & \\
\text{subject to} & \sum_{j=1}^{n} a_{ij} x_j \le b_i & (i = 1, 2, \ldots, m), \\
 & x_j \in \{0, 1\} & (j = 1, 2, \ldots, n)
\end{array}
$$

0-1 Knapsack problem

- Knapsack problem is the simplest capital budgeting problem with only one resource.
- You have n items to choose from to put into your knapsack.
- Item j has weight a_j, and it has value c_j.
- The maximum weight your knapsack can hold is b.

Formulate the knapsack problem:

$$
\begin{array}{ll}
\text{Maximize} & \sum_{j=1}^{n} c_j x_j, \\
\text{subject to} & \sum_{j=1}^{n} a_j x_j \leq b, \\
 & x_j \in \{0, 1\} \qquad (j = 1, 2, \ldots, n).
\end{array}
$$

Warehouse Location

A manager must decide which of n warehouses to use for meeting the demand of m customers for a good. The decisions to be made are which warehouses to operate and how much to ship from any warehouse to any customer. Let

$$y_i = \begin{cases} 1 & \text{if warehouse } i \text{ is opened,} \\ 0 & \text{if warehouse } i \text{ is not opened;} \end{cases}$$

$x_{ij} =$ Amount to be sent from warehouse i to customer j.

The relevant costs are:

$f_i =$ Fixed operating cost for warehouse i, if opened
$c_{ij} =$ Per-unit operating cost at warehouse i plus the transportation cost for shipping from warehouse i to customer j.

There are two types of constraints for the model:

1. the demand d_j of each customer must be filled from the warehouse; and
2. goods can be shipped from a warehouse only if it is opened.

Suppose you knew which warehouses were open.

Let $S =$ set of open warehouses.

- $x_{ij} =$ demand satisfied for customer j at warehouse i
- $y_i = 1$ for i in S.
  $y_i = 0$ for i not in S.

Min $$\sum_{i,j} c_{ij} x_{ij} + \sum_{i \in S} f_i$$

Subject to:

- customers get their demand satisfied

$$\sum_i x_{ij} = d_j$$

- no shipments are made from an empty warehouse

$$\begin{array}{ll} x_{ij} \leq d_j & \text{if } y_i = 1 \\ x_{ij} = 0 & \text{if } y_i = 0 \end{array}$$

and $x \geq 0$

More on warehouse location

- x_{ij} = demand satisfied for customer j at warehouse i
- $y_i = 1$ for warehouse i is opened.
  $y_i = 0$ otherwise.

$$\text{Min} \quad \sum_{i,j} c_{ij} x_{ij} + \sum_i f_i y_i$$

Subject to:

- customers get their demand satisfied

$$\sum_i x_{ij} = d_j$$

- each warehouse is either opened or it is not (no partial openings)

$$0 \le y_i \le 1$$
$$y_i \text{ integral for all } i$$

- no shipments are made from an empty warehouse

$$x_{ij} \le d_j y_i \text{ for all } i, j$$

$$\text{and } x \ge 0$$

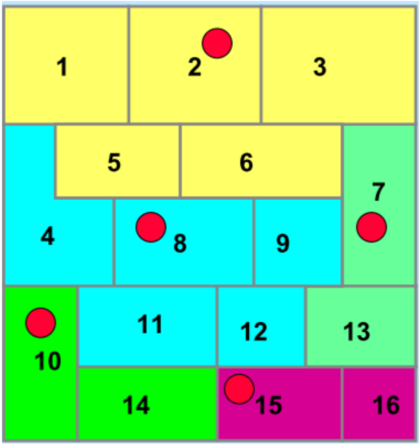
The above is a core subproblem in supply chain management, and it can be enriched

- more complex distribution system
- capacity constraints
- non-linear transportation costs
- delivery time restriction
- multiple products
- business rules
- and more

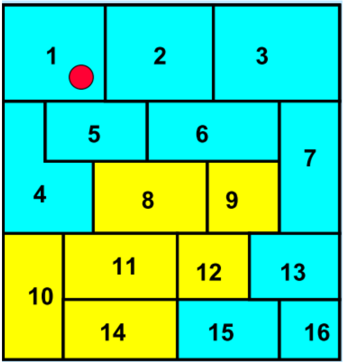
Fire Station Problem

- Locate the fire stations so that each district has a fire station in it, or next to it.
- Minimize the number of fire stations needed

Here is one feasible solution with five fire stations.



Representation as Set Covering Problem



Set no.	Set
1	{1,2,4,5}
2	{1,2,3,5,6}
3	{2,3,6,7}
⋮	⋮
16	{13, 15, 16}

Representation as an IP

Decision variables:

- which district to choose
- which sets to choose

Constraints:

- each district has a fire station or is next to one
- each element gets covered

Set no.	Set
1	{1,2,4,5}
2	{1,2,3,5,6}
3	{2,3,6,7}
⋮	⋮
16	{13, 15, 16}

Representation as an IP

$x_j = 1$ if set j is selected
$x_j = 0$ otherwise

Min $x_1 + x_2 + \ldots + x_{16}$

s.t. $x_1 + x_2 + x_4 + x_5 \geq 1$

$x_1 + x_2 + x_3 + x_5 + x_6 \geq 1$

$\vdots \quad \vdots$

$x_{13} + x_{15} + x_{16} \geq 1$

$x_j \in \{0, 1\}$ for each j.

Set no.	Set
1	{1,2,4,5}
2	{1,2,3,5,6}
3	{2,3,6,7}
⋮	⋮
16	{13, 15, 16}

On Covering Problem

"Covering problems" model a number of applied situations.

- Assigning pilots and stewards to planes
- Assigning police cars, ambulances, and other city personnel
- package delivery, oil delivery, etc.

Summary on Integer Programming

- Dramatically improves the modelling capability
  - Integral quantities
  - Logical constraints
  - Modeling fixed charges
  - Classical problems in capital budgeting and location
- Not as easy to model
- Not as easy to solve