Lecture 20 on Nov. 28 2013

We have seen the applications of the simplest Cauchy theorem in the above lectures. Today we are going to consider its more general version.

Definition 0.1. *A curve γ in Ω is said to be homogeneous to a point in Ω if γ can be deformed continuously to the point. Analytically we have a two variable functions $\gamma(t,s)$ from the rectangle $[0,1]\times[0,1]$ to Ω so that γ is continuous with respect to both t variable and s variable. Moreover $\gamma(t,0)$ is a parametrization of the curve γ and $\gamma(t,1)$ is constantly equal to the given point in Ω.*

One may refer to the figure 1 to take a glance on the concept introduced above. In fact in figure 1, Ω_2 is the larger domain and Ω_1 is the smaller domain inside Ω_2. Our Ω is the domain in Ω_2 without the domain Ω_1. For γ_1 no matter how you deform γ_1 to a point in Ω, you will alway intersect some points in Ω_1. But for γ_2 we can do so. The difference of these two curves are the follows. For γ_1, it enclose an interior region and Ω_1 is included in the region. While for γ_2, Ω_1 is outside the region enclosed by γ_2. In the following, we are going to show that Cauchy's theorem still holds for curves with the same type of γ_2.

Given Γ_1 with positive orientation (see figure 2), we choose another curve Γ_2 which is quite close to Γ_1. We seperate the region between Γ_1 and Γ_2 into a lot of small boxes. The size of each box is small enough so that for each small box, we can find a disk to cover it and f is analytic in the disk. Now we zoom out the box A and box B and choose the contour as what is shown in figure 3. Clearly by simple Cauchy thoerem, we know that

$$\int_{I_1+I_2+I_3+I_4} f(z)\,\mathrm{d}z = 0$$

where f is an analytic function in a domain containing Γ_1. Moreover we also have

$$\int_{J_1+J_2+J_3+J_4} f(z)\,\mathrm{d}z = 0$$

Pay attention that I_4 and J_2 are interface between A and B but they have different direction. So the integration on I_4 and J_2 can be cancelled with each other. Therefore if we add the above two equalities, we get

$$\int_{I_1+I_2+I_3+J_3+J_4+J_1} f(z)\,\mathrm{d}z = 0.$$

In this new contour, the interface between A and B disappear. The same technique can be applied to the remaining boxes and show that

$$\int_{\Gamma_1-\Gamma_2} f(z)\,\mathrm{d}z = 0. \tag{0.1}$$

notice here Γ_2 is chosen to be positively oriented. From Figure 3, we see that the curve I_3+J_3 have different orientation from I_1+J_1. Therefore after cancellation of interfaces, the outer curve should be Γ_1 and has the same orientation as Γ_1 but the interior curve coincide with Γ_2 but have different orientation as we choose for the Γ_2. That is why we have a negative sign in front of Γ_2 in (0.1). Rewriting (0.1), we obtain

$$\int_{\Gamma_1} f(z)\,\mathrm{d}z = \int_{\Gamma_2} f(z)\,\mathrm{d}z.$$

If Γ_1 can be deformed to a point P (see Figure 2) and f is analytic on a disk around P, then we know that

$$\int_{\gamma} f(z)\,\mathrm{d}z = 0.$$

Therefore we further show that

$$\int_{\Gamma_1} f(z)\,\mathrm{d}z = 0.$$

Summarizing all the arguments above, we have

Theorem 0.2. *if f is analytic in a domain Ω and γ is a closed curve homogeneous to a point in Ω, then*

$$\int_\gamma f(z)\,\mathrm{d}z = 0.$$

A straightforward application of Theorem 0.2 is the so-called Laurent series. Given an annulus shown as in Figure 4, z_0 is the center. The outer circle has radius r_2 and interior circle has radius r_1. z is an arbitrary point on the annulus. If $f(\zeta)$ is analytic on the annulus, then by removability of singularities, $(f(\zeta)-f(z))/(\zeta-z)$ is also analytic in the annulus with resepct to the variable ζ. Choosing the contour $I_1+I_2+I_3+I_4$, it is homogeneous to a point in the annulus, therefore we have by Theorem 0.2 that

$$\int_{I_1+I_2+I_3+I_4} \frac{f(\zeta)-f(z)}{\zeta-z}\,\mathrm{d}\zeta = 0.$$

I_2 and $_4$ can be cancelled with each other since they have different direction, therefore we obtain from the above equality that

$$f(z)\int_{I_1+I_3} \frac{1}{\zeta-z}\,\mathrm{d}\zeta = \int_{I_1+I_3} \frac{f(\zeta)}{\zeta-z}\,\mathrm{d}\zeta. \tag{0.2}$$

Noting that the index of z with resepct to I_1 equals to 1 and the index of z with respect to I_3 is 0, therefore the left-hand side of (0.2) equals to $2\pi i f(z)$. furthermore (0.2) can be rewritten as

$$f(z) = \frac{1}{2\pi i}\int_{I_1} \frac{f(\zeta)}{\zeta-z}\,\mathrm{d}\zeta + \frac{1}{2\pi i}\int_{I_3} \frac{f(\zeta)}{\zeta-z}\,\mathrm{d}\zeta. \tag{0.3}$$

Now we deal with the integraion on I_1 on the right-hand side of (0.3). clearly

$$\begin{aligned}
\frac{1}{2\pi i}\int_{I_1} \frac{f(\zeta)}{\zeta-z}\,\mathrm{d}\zeta &= \frac{1}{2\pi i}\int_{I_1} \frac{f(\zeta)}{\zeta-z_0+z_0-z}\,\mathrm{d}\zeta \\
&= \frac{1}{2\pi i}\int_{I_1} \frac{f(\zeta)}{\zeta-z_0}\,\frac{1}{1-\frac{z-z_0}{\zeta-z_0}}\,\mathrm{d}\zeta.
\end{aligned}$$

Noticing that on I_1, $|\zeta-z_0|>|z-z_0|$, therefore it holds by geometric series that

$$\frac{1}{2\pi i}\int_{I_1} \frac{f(\zeta)}{\zeta-z}\,\mathrm{d}\zeta = \frac{1}{2\pi i}\int_{I_1} \frac{f(\zeta)}{\zeta-z_0}\sum_{k=0}^{\infty}\left(\frac{z-z_0}{\zeta-z_0}\right)^k \mathrm{d}\zeta \tag{0.4}$$

$$\tag{0.5}$$

$$= \sum_{k=0}^{\infty}\left(\frac{1}{2\pi i}\int_{I_1} \frac{f(\zeta)}{(\zeta-z_0)^{k+1}}\,\mathrm{d}\zeta\right)(z-z_0)^k. \tag{0.6}$$

As for the integration on I_3, similarly we have

$$\begin{aligned}
\frac{1}{2\pi i}\int_{I_3} \frac{f(\zeta)}{\zeta-z}\,\mathrm{d}\zeta &= \frac{1}{2\pi i}\int_{I_3} \frac{f(\zeta)}{\zeta-z_0+z_0-z}\,\mathrm{d}\zeta \\
&= -\frac{1}{2\pi i}\int_{I_3} \frac{f(\zeta)}{z-z_0}\,\frac{1}{1-\frac{\zeta-z_0}{z-z_0}}\,\mathrm{d}\zeta \\
&= \frac{1}{2\pi i}\int_{-I_3} \frac{f(\zeta)}{z-z_0}\sum_{k=0}^{\infty}\left(\frac{\zeta-z_0}{z-z_0}\right)^k \mathrm{d}\zeta \\
&= \sum_{k=0}^{\infty}\left(\frac{1}{2\pi i}\int_{-I_3} f(\zeta)\,(\zeta-z_0)^k\,\mathrm{d}\zeta\right)(z-z_0)^{-(k+1)}.
\end{aligned}$$

summarizing the above arguments, we know that

Theorem 0.3. *if f is analytic on the annulus with outer circle I_1 and inner circle I_3 (see figure 4), then f can be expanded by*

$$f(z) = \sum_{k=-\infty}^{\infty} A_k (z - z_0)^k,$$

where

$$A_k = \frac{1}{2\pi i} \int_\Gamma \frac{f(\zeta)}{(\zeta - z_0)^{k+1}} \,\mathrm{d}\zeta.$$

Here if $k = 0, 1, 2, \ldots$, then Γ in A_k is the positively oriented outer circle I_1. If $k = -1, -2, \ldots$, then Γ is the positively oriented inner circle I_3.

Using Theorem 0.3, we see that

$$f(z) = \sum_{k=-2}^{-\infty} A_k (z - z_0)^k + \sum_{k=0}^{\infty} A_k (z - z_0)^k + \frac{A_{-1}}{z - z_0}.$$

all functions on the right-hand side above has anti-derivatives except the function

$$\frac{A_{-1}}{z - z_0}.$$

Therefore given a closed curve γ in the annulus, we can easily show that

$$\frac{1}{2\pi i} \int_\gamma f(z) \,\mathrm{d}z = \frac{1}{2\pi i} \int_\gamma \frac{A_{-1}}{z - z_0} \,\mathrm{d}z = A_{-1} n(\gamma, z_0). \tag{0.7}$$

From the above calculations, we see that A_{-1} is of most important to us comparing to the other coefficients. So we give a special name for it.

Definition 0.4. *We call A_{-1} the residue of a given function f at z_0, denoted by* $\mathrm{Res}(f, z_0)$*. The expansion in Theorem 0.3 is called Laurent series.*

Before moving forward, let us study the uniqueness of the expansion in Theorem 0.3 and a little bit generalization of (0.7).

Uniqueness of Expansion Suppose that there is another expansion of f on annulus, say

$$f(z) = \sum_{k=-\infty}^{\infty} B_k (z - z_0)^k,$$

then clearly we have

$$\frac{1}{2\pi i} \int_{I_1} f(z) = \frac{1}{2\pi i} \int_{I_1} \frac{A_{-1}}{z - z_0} = \frac{1}{2\pi i} \int_{I_1} \frac{B_{-1}}{z - z_0} = A_{-1} = B_{-1}.$$

Multiply $f(z)$ by $z - z_0$ and applying the same calculations, we know that

$$\frac{1}{2\pi i} \int_{I_1} (z - z_0) f(z) = \frac{1}{2\pi i} \int_{I_1} \frac{A_{-2}}{z - z_0} = \frac{1}{2\pi i} \int_{I_1} \frac{B_{-2}}{z - z_0} = A_{-2} = B_{-2}.$$

Inductively we know that for any k, it holds

$$\frac{1}{2\pi i} \int_{I_1} (z - z_0)^k f(z) = \frac{1}{2\pi i} \int_{I_1} \frac{A_{-(k+1)}}{z - z_0} = \frac{1}{2\pi i} \int_{I_1} \frac{B_{-(k+1)}}{z - z_0} = A_{-(k+1)} = B_{-(k+1)}.$$

Therefore we have

Theorem 0.5. *If on a annulus f can be written as*

$$f(z) = \sum_{k=-\infty}^{\infty} B_k (z - z_0)^k,$$

then it must be the Laurent series of f.

Generalization of (0.7) The generalization of (0.7) in the following is the so-called Residue theorem

Theorem 0.6. *Given a closed curve γ positively oriented (see figure 5) and letting Ω is the region enclosed by γ, if f is analytic in Ω except finitely many singularities $\{z_1, ..., z_n\}$, then*

$$\frac{1}{2\pi i} \int_\gamma f(z)\,\mathrm{d}z = \sum_{k=1}^{n} \mathrm{Res}(f, z_j).$$

The proof of this theorem is simple. using the contour in figure 5, we can easily show that

$$\frac{1}{2\pi i} \int_\gamma f(z)\,\mathrm{d}z = \sum_{k=1}^{n} \frac{1}{2\pi i} \int_{\gamma_k} f(z)\,\mathrm{d}z. \tag{0.8}$$

Here in (0.8), we used the general Cauchy theorem. Then apply (0.7) to the right-hand side above, the proof of Theorem 0.6 follows.

In light of the above arguments, we know that the most important thing in the evaluating of contour integral for a complex function is to find out its residue. Here we give a method to search residues of some special functions.

Case 1. In this case we assume z_0 is a singularity of f and moreover

$$\lim_{z \to z_0} (z - z_0) f(z) = c,$$

where c is constant. We claim that in this case c equals to the residue of f at z_0. In fact, we consider the function

$$g(z) = f(z) - \frac{c}{z - z_0}.$$

by the assumption above, one can easily show that

$$\lim_{z \to z_0} (z - z_0) g(z) = 0.$$

Therefore applying the removability of singularity to g, we know that g is analytic at z_0. In other words

$$f(z) = \frac{c}{z - z_0} + g(z),$$

where g is analytic at z_0. Clearly g can be expanded by Taylor series, Therefore by the uniqueness theorem 0.5, we know that

$$f(z) = \frac{c}{z - z_0} + \text{Taylor Series of } g.$$

Clearly c is the coefficient in front of $\frac{1}{z - z_0}$. That is the residue of f at z_0.

Example 1. suppose that $a \neq b$ are two complex numbers, then

$$\frac{e^z}{(z-a)(z-b)}$$

has two singularities, a and b. Since

$$\lim_{z\to a}\frac{e^z}{(z-a)(z-b)}(z-a)=\lim_{z\to a}\frac{e^z}{z-b}=\frac{e^a}{a-b}.$$

Therefore we have

$$\text{Res}\left(\frac{e^z}{(z-a)(z-b)},a\right)=\frac{e^a}{a-b}.$$

Example 2. consider $1/\sin z$. This function has singularities at $k\pi$ where k are all integers. by L'Hospitale rule, we know that

$$\lim_{z\to k\pi}\frac{z-k\pi}{\sin z}=(-1)^k,$$

Therefore it holds

$$\text{Res}(\frac{1}{\sin z},k\pi)=(-1)^k.$$

Case 2. The functions in case 2 are powers of all functions in case 1. Since the functions in case 1 can be written as

$$f(z)=\frac{c}{z-z_0}+g(z)$$

where $g(z)$ is analytic at z_0. Therefore

$$(f(z))^n=\left(\frac{c}{z-z_0}+g(z)\right)^n$$

Using Binomial formula, we know that the higher order of the pole z_0 must be n. So in order to get the coefficient of $(z-z_0)^{-(n-1)}$, we just need move $c^n/(z-z_0)^n$ to the left and calculuate the limit

$$\lim_{z\to z_0}\left((f(z))^n-\frac{c^n}{(z-z_0)^n}\right)(z-z_0)^{n-1}.$$

Then can we get the coefficient of $A_{-(n-1)}$ from the above limit. To get $A_{-(n-2)}$ we just need move $A_{-(n-1)}/(z-z_0)^{n-1}$ to the left and calculuate

$$\lim_{z\to z_0}\left((f(z))^n-\frac{c^n}{(z-z_0)^n}-\frac{A_{-(n-1)}}{(z-z_0)^{n-1}}\right)(z-z_0)^{n-2}.$$

Inductively we can find out the coefficient A_{-1} in finite steps.

Example 3. $1/\sin^2 z$. We know that

$$\frac{1}{\sin^2 z}=\left(\frac{1}{z}+g(z)\right)^2,$$

in a neighborhood of $z_0=0$. Therefore $z_0=0$ is a pole of $1/\sin^2 z$ with order 2. To get A_{-1} at $z_0=0$, we just need calculate

$$\lim_{z\to 0}z\left(\frac{1}{\sin^2 z}-\frac{1}{z^2}\right)$$

Finally one can show that the above limit is 0.