Lecture 18-19 on Nov. 25 2013

These two lectures are devoted to studying the integral

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}\,\mathrm{d}z,$$

where γ is a simple curve enclosing the region Ω. The readers are referred to the figure 1 in the pdf file *the graph of lecture 18-19*. In fact when $f(z) = z$, we know that this integral gives us the so called index of 0 with respect to the curve γ. Since our curve γ is simple, the index is either 1 or -1 for all points in Ω. In the following arguments, we always assume that γ is positively oriented so that the index of all points inside Ω equal to 1 with respect to the curve γ. We also assume that $f(z)$ in the study is not a constant function and moreover $f \neq 0$ **on the curve** γ. With this assumption, we know that $f(z)$ can be factorized by

$$f(z) = (z - z_1)(z - z_2)...(z - z_n)g(z), \tag{0.1}$$

where $g(z)$ is analytic in Ω and $g(z) \neq 0$ for all z in Ω. From (0.1), we see that $z_1, ...z_n$ are n zeros of f. According to Theorem 0.7 in lecture note 17, we know that f can have only finitely many zeros in Ω. Therefore by removability of singularity theorem, one can easily show that (0.1) holds.

By (0.1), we calculate

$$\frac{f'(z)}{f(z)} = \frac{1}{z - z_1} + ... + \frac{1}{z - z_n} + \frac{g'(z)}{g(z)}.$$

Therefore by the definition of index and Cauchy-Gousat theorem, one can easily show that

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}\,\mathrm{d}z = n(z_1, \gamma) + ... + n(z_n, \gamma) = 1 + ... + 1 = n.$$

Therefore if γ is positively oriented,

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}\,\mathrm{d}z = \text{Total number of zeros of } f \text{ in } \Omega. \tag{0.2}$$

We can make two generalizations of (0.2).

First Generalization: Assume $F(z) = f(z) - a$ where a is a complex number so that $f \neq a$ on γ. By (0.2), we have

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z) - a} = \frac{1}{2\pi i}\int_\gamma \frac{F'(z)}{F(z)} = \text{Total number of zeros of } F \text{ in } \Omega.$$

Clearly the zeors of F in Ω are all solutions of the equation $f = a$ in Ω. Therefore we have

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z) - a} = \text{Total number of solutions of the equation } f = a \text{ in } \Omega. \tag{0.3}$$

Second Generalization: Assume

$$f(z) = \frac{F(z)}{G(z)},$$

where $F(z)$ and $G(z)$ are two analytic functions in Ω. Suppose that both F and G have no zeros on γ. By trivial calcuations, we know that

$$\frac{f'(z)}{f(z)} = \frac{F'(z)}{F(z)} - \frac{G'(z)}{G(z)}.$$

Applying (0.2), we show that

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)} = (\text{Total number of zeros of } F \text{ in } \Omega) - (\text{Total number of zeros of } G \text{ in } \Omega). \tag{0.4}$$

Now we are going to explore some applications of these two generalizations.

Application of the First Generalization. Assume $f(z_0) = a$ where z_0 is a point in Ω. By the isolation of zeros, we can shrink γ a little bit so that in Ω there is only one solution of the equation $f(z) = a$. That is z_0. Therefore we know by (0.3) that

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)-a} = \text{Total number of solutions of the equation } f = a.$$

Must the right-hand side of the above equality equal to 1 since we have only one solution of $f = a$ in Ω ? Let us take a look at the Taylor expansion of f near z_0. By Taylor expansion, we know that

$$f(z) = f(z_0) + f'(z_0)(z-z_0) + ... + \frac{f^{(k)}(z_0)}{k!}(z-z_0)^k + g_{k+1}(z)(z-z_0)^{k+1}.$$

Since we assume that f is not a constant, there must be a k so that for all $i < k$ and $i > 0$, the derivatives $f^{(i)}(z_0) = 0$ but $f^{(k)}(z_0) \neq 0$. Therefore it holds

$$f(z) = f(z_0) + (z-z_0)^k\left(\frac{f^{(k)}(z_0)}{k!} + g_{k+1}(z-z_0)\right)$$

Set

$$h_{k+1} = \frac{f^{(k)}(z_0)}{k!} + g_{k+1}(z-z_0).$$

clearly when z is close to z_0, $h_{k+1}(z) \neq 0$. Therefore we know that

$$f(z) - a = (z-z_0)^k h_{k+1}. \tag{0.5}$$

Moreover

$$\frac{f'(z)}{f(z)-a} = \frac{k}{z-z_0} + \frac{h'_{k+1}}{h_{k+1}}.$$

Now if we require γ is sufficiently close to z_0, then

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)-a} = k.$$

This k could be different from 1 since from (0.5), even though we have just one solution of $f = a$, but this solution z_0 could be repeated by k times. In the future, we call k the multicity of z_0 with respect to the equation $f = a$. With the above arguments, we know that

$$\frac{1}{2\pi i}\int_\gamma \frac{f'}{f-a}$$

counts the total number of solutions of $f = a$. Repeated solutions will also be counted.

Now we fix γ sufficiently close to z_0 so that z_0 is the isolated solution of the equation $f = a$. If we assume b sufficiently close to a, then clearly

$$\frac{1}{2\pi i}\int_\gamma \frac{f'}{f-b}$$

is sufficiently close to

$$\frac{1}{2\pi i}\int_\gamma \frac{f'}{f-a}$$

But these two numbers are all integers. So we know that

$$\frac{1}{2\pi i}\int_\gamma \frac{f'}{f-b} = \frac{1}{2\pi i}\int_\gamma \frac{f'}{f-a}, \qquad \text{provided that } b \text{ is sufficiently close to } a. \tag{0.6}$$

With (0.6), we can prove the following maximum mudulus theorem

Theorem 0.1 (Maximum Modulus Theorem)**.** *If f is not a constant function on Ω, then the maximum value of $|f(z)|$ can only be attained on the boundary of Ω. That is γ.*

Proof. Choose z_0 in Ω and assume $|f(z_0)|$ attains the maximum value of $|f(z)|$ in Ω. Clearly

$$f(z_0) \neq 0.$$

otherwise, $f(z) = 0$ for all z in Ω. Using Taylor expansion, we know that

$$f(z) = f(z_0) + (z - z_0)^k g(z), \tag{0.7}$$

where $g(z) \neq 0$ in $|z - z_0| < \epsilon$. Here ϵ is a small positive constant. The equation

$$(z - z_0)^k g(z) = 0$$

has k repeated solutions in $|z - z_0| < \epsilon$. Therefore by (0.6), we know that

$$(z - z_0)^k g(z) = \delta f(z_0),$$

also has k solutions in $|z - z_0| < \epsilon$. Here δ is a positive number sufficiently small. Fixing z_* in $|z - z_0| < \epsilon$ so that $(z_* - z_0)^k g(z_*) = \delta f(z_0)$. Therefore we know by (0.7) that

$$f(z_*) = f(z_0) + (z_* - z_0)^k g(z_*) = f(z_0) + \delta f(z_0) = (1 + \delta) f(z_0).$$

therefore we know that $|f(z_*)| = (1 + \delta)|f(z_0)| > |f(z_0)|$. A contradiction. So the maximum modulus of f can never be attained in Ω if f is not a constant. □

Now we see how to apply Theorem 0.1.

Example 1. The lemma of Schwartz.

Proposition 0.2. *Assume f is analytic in $|z| < 1$. $|f(z)| \leq 1$ for all z in $|z| < 1$. Furthermore we suppose that $f(0) = 0$. Then with the above assumption, it holds*

$$|f(z)| \leq |z|, \qquad \text{for all } z \text{ in } |z| < 1.$$

If $|f(z_*)| = |z_*|$ for some z_* in $|z| < 1$, then $f(z) = cz$ for all z in $|z| < 1$. Here c is a constant with $|c| = 1$.

Proof. Step 1. define $g(z) = f(z)/z$. This function is analytic in $0 < |z| < 1$. By Removability of singularity, we know that g is analytic in $|z| < 1$;

Step 2. Choosing an arbitrary $r < 1$ and apply the maximum modulus theorem to g with the $\Omega = \{|z| \leq r\}$. Clearly we know that

$$\left|\frac{f(z)}{z}\right| \leq \max_{w \text{ on } |z| = r} \left|\frac{f(w)}{w}\right| \leq \frac{1}{r} \longrightarrow 1, \qquad \text{as } r \to 1.$$

This shows that $|f(z)| \leq |z|$;

Step 3. If there is z_* so that $|f(z_*)| = |z_*|$, then by Theorem 0.1, $f(z)/z$ must be a constant. Therefore $f(z) = cz$. clearly $|c| = 1$ since $|f(z_*)| = |z_*|$. □

Application of the Second Generalization. To apply the second generalization, we need take a close look at the integral

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}\,\mathrm{d}z.$$

Assume $z(t)$ is one parametrization of γ with t defined on $[a,b]$. Then

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}\,\mathrm{d}z = \frac{1}{2\pi i}\int_a^b \frac{f'(z(t))}{f(z(t))}\,z'(t)\mathrm{d}t = \frac{1}{2\pi i}\int_a^b \frac{(f(z(t)))'}{f(z(t))}\,\mathrm{d}t.$$

In the second inequality, the chain rule is applied. Assume $w(t) = f(z(t))$. Therefore the above integral can be rewritten as

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}\,\mathrm{d}z = \frac{1}{2\pi i}\int_a^b \frac{w'(t)}{w(t)}\,\mathrm{d}t = \frac{1}{2\pi i}\int_{\Gamma=f(\gamma)} \frac{1}{w}\,\mathrm{d}w = n(0,\Gamma).$$

Combing the above calculations with (0.4), we know that

Proposition 0.3 (Argument Principle)**.** *if $f = F/G$, then*

$$n(0,\Gamma) = \textit{(Total number of zeros of } F \textit{ in } \Omega) - \textit{(Total number of zeros of } G \textit{ in } \Omega).$$

Here $\Gamma = f(\gamma)$.

Proposition 0.3 has a straightforward corollary.

Theorem 0.4 (Rouche's theorem)**.** *If $|g - f| < |f|$ on γ, then f and g have the same number of zeros in Ω.*

Proof. Clearly by the above assume $f \neq 0$ on γ and moreover $g \neq 0$ on γ too. Consider g/f. By Proposition 0.3, we know that

$$n(0,(g/f)(\gamma)) = (\text{Total number of zeros of } g \text{ in } \Omega) - (\text{Total number of zeros of } f \text{ in } \Omega). \tag{0.8}$$

According to our assumption,

$$|(g/f)(z) - 1| < 1, \qquad \text{for all } z \text{ on } \gamma.$$

In other words, $(g/f)(\gamma)$ is inside the ball $|w-1| < 1$. But 0 is not in this ball, therefore we conclude that $n(0,\Gamma) = 0$. This implies that

$$(\text{Total number of zeros of } g \text{ in } \Omega) = (\text{Total number of zeros of } f \text{ in } \Omega).$$

□

Example 2. How many roots of $g(z) = z^8 - 8z^6 + z^3 + z^2 + 2$ lie inside the unit disk $|z| < 1$.

Solution: Letting $f(z) = -8z^6$, we know that

$$|g(z) - f(z)| = |z^8 + z^3 + z^2 + 2| \leq 5, \qquad \text{on } |z| = 1.$$

But $|f| = 8$ on $|z| = 1$. Therefore we have $|g - f| < |f|$ on $|z| = 1$. By Rouche's theorem, there are 6 roots of g inside $|z| < 1$ since $f(z) = 0$ has six roots in $|z| < 1$. Notice here f in fact has six repeated roots. The multicity has to be counted.

Example 3. How many roots of the polynomial $g(z) = z^4 + 3z^2 + 8z + 2$ lie on the right-half plane.

Solution: Construct the contour γ_R by the following way. The first part of γ_R contains all points on the pure imaginary line between $-Ri$ and Ri. The second part contains all points on the right-half of the circle $|z| = R$. We choose positive orientation of γ_R and denote by I the set of points on the first part. and II the set of points on the second part. The readers are referred to the figure 2 in the graph file. By the argument principle in Proposition 0.3, we know that the total number of zeros of g equals to $n(0, g(\gamma_R))$ when R is large enough.

the image of I under the mapping g. Assume I is parametrized by ti where t is the parameter from R to $-R$. Plugging into g, we know that

$$g(ti) = (t-1)(t+1)(t-\sqrt{2})(t+\sqrt{2}) + 8ti.$$

The image of I under the mapping g is shown in figure 3. Clearly the total change of arguments equals to

$$-2\text{arctan}\left(\frac{8R}{(R-1)(R+1)(R+\sqrt{2})(R-\sqrt{2})}\right) \longrightarrow 0, \qquad \text{as } R \to \infty.$$

Therefore while R is large enough, the change of arguments on part I is very small.

the image of II under the mapping g. Assume II is parametrizaed by $Re^{i\theta}$ where θ runs from $-\pi/2$ to $\pi/2$. Therefore

$$g(Re^{i\theta}) = R^4 e^{i4\theta} + 3R^2 e^{i2\theta} + 8Re^{i\theta} + 2 = R^4\left(e^{i4\theta} + 3R^{-2}e^{i2\theta} + 8R^{-3}e^{i\theta} + 2R^{-4}\right).$$

Noting that $e^{i4\theta} + 3R^{-2}e^{i2\theta} + 8R^{-3}e^{i\theta} + 2R^{-4}$ is a small perturbation of $e^{i4\theta}$ while $R \to \infty$. Therefore the total change of argument from part II equals to 4π while $R \to \infty$. Therefore the total change of argument along $g(\gamma_R)$ equals to 4π while $R \to \infty$. The index $n(0, g(\gamma_R)) = 4\pi/2\pi = 2$ while R is large enough. So there are 2 roots of g on the right-half plane.