Solution 9

Section 8.2

1. Let $f_n(x) = \frac{x^n}{1+x^n}$ and f_n converges to f.
   Then f must be the pointwise limit of f_n,
   $$f(x) = \begin{cases} 0, & x \in [0,1) \\ \frac{1}{2}, & x = 1 \\ 1, & x \in (1,2] \end{cases}$$
   Obviously, f is not continuous, and hence f_n doesn't converge uniformly to f.

4. Let $\varepsilon > 0$ be given. Then by the uniform convergence of $\{f_n\}$ to f on I, since each f_n is continuous on I, we have that f is continuous on I. Hence there exists $\delta > 0$ such that whenever $x \in I$ and $|x - x_0| < \delta$, we have
   $$|f(x) - f(x_0)| < \frac{\varepsilon}{2}.$$
   Now since $\{x_n\}$ converges to x_0, for the δ chosen above, there exists $N_1 \in \mathbb{N}$ such that whenever $n \geq N$, we have
   $$|x_n - x_0| < \delta.$$
   Of course $x_n \in I$ for all n, so whenever $n \geq N_1$, we have
   $$|f(x_n) - f(x_0)| < \frac{\varepsilon}{2}. \tag{1}$$
   Also, by the uniform convergence of f_n to f on I, there exists $N_2 \in \mathbb{N}$ such that whenever $n \geq N_2$ and $x \in I$, we have
   $$|f_n(x) - f(x)| < \frac{\varepsilon}{2};$$
   in particular, whenever $n \geq N_2$, since $x_n \in I$ we have
   $$|f_n(x_n) - f(x_n)| < \frac{\varepsilon}{2}. \tag{2}$$
   Hence combining (1) and (2), we see that if we take $N = \max\{N_1, N_2\}$, then whenever $n \geq N$, we have
   $$|f_n(x_n) - f(x_0)| \leq |f_n(x_n) - f(x_n)| + |f(x_n) - f(x_0)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$
   This proves that
   $$\lim_{n\to\infty} f_n(x_n) = f(x_0).$$
   Remark. The proof may look complicated, but the idea is simple: when n is big,
   $$f(x_n) \simeq f(x_0)$$
   by continuity of f, and
   $$f_n(x_n) \simeq f(x_n)$$
   by uniform convergence of f_n to f. Hence when n is big,
   $$f_n(x_n) \simeq f(x_0),$$
   and this is the proof above.

5. Let $\varepsilon > 0$ be given. Then by uniform continuity of f, there exists $\delta > 0$ such that whenever $x, y \in \mathbb{R}$ satisfies $|x - y| < \delta$, we have
$$|f(x) - f(y)| < \varepsilon; \tag{3}$$
in particular, if $N \in \mathbb{N}$ is such that $N > \frac{1}{\delta}$, then for any $n \geq N$ and any $x \in \mathbb{R}$, we have
$$\left| \left(x + \frac{1}{n} \right) - x \right| = \frac{1}{n} \leq \frac{1}{N} < \delta,$$
which implies, by (3), that
$$|f_n(x) - f(x)| = \left| f\left(x + \frac{1}{n} \right) - f(x) \right| < \varepsilon.$$
This proves the uniform convergence of f_n to f on $\mathbb{R}$.

6.
$$f(x) = \begin{cases} 1, & x = 0 \\ 0, & x \in (0, 1] \end{cases}$$
Obviously, f is not continuous, and hence f_n doesn't converge uniformly to f.

7. Since each f_n is bounded on A, for each $n \in \mathbb{N}$, there exists a constant M_n such that
$$|f_n(x)| \leq M_n$$
for all $x \in A$. Now take $\varepsilon = 1$. Then by uniform convergence of f_n to f on A, there exists $N \in \mathbb{N}$ such that whenever $n \geq N$ and $x \in A$, we have
$$|f_n(x) - f(x)| < 1;$$
in particular, for any $x \in A$, we have
$$|f(x)| \leq |f(x) - f_N(x)| + |f_N(x)| < 1 + M_N.$$
This proves that f is bounded on A.

Remark. One should observe that the proof is simple if we use sup-norm: Indeed since f_n is bounded by M_n on A, we have
$$\|f_n\|_A \leq M_n.$$
But since f_n converges to f uniformly on A, there exists $N \in \mathbb{N}$ such that for any $n \geq N$, we have
$$\|f_n - f\|_A < 1.$$
Hence using the triangle inequality for the sup-norm, we get
$$\|f\|_A \leq \|f - f_N\|_A + \|f_N\|_A < 1 + M_N,$$
which says that f is bounded on A.

9. Let $f(x) := \lim f_n(x) = \lim x^n/n = 0$, since $x \in [0, 1]$. Hence f is differentiable.
Moreover, $\| f_n - f \|_{[0,1]} = \sup\limits_{x \in [0,1]} x^n/n \leq 1/n \to 0$.

Thus f_n converges uniformly to the differentiable function f on $[0, 1]$.

Now $f_n'(x) = x^{n-1}$ converges pointwisely to $g(x) := \lim f_n'(x) = \lim x^{n-1} = \begin{cases} 1, & x = 1 \\ 0, & \text{otherwise} \end{cases}$

But $f'(x) = 0 \ \Rightarrow \ f'(1) = 0 \neq 1 = g(1)$.

11. Let $F_n(x) = \int_a^x f_n'(t)dt$ and $F(x) = \int_a^x g(t)dt$. Note that g is continuous, therefore, F is well defined.

    Claim: F_n converges uniformly to F.

    proof of the Claim: Given $\varepsilon > 0$, since f_n' converges uniformly to g, then there exists $N > 0$ such that for any $n > N$, we have

$$|f_n'(t) - g(t)| < \varepsilon$$

    for any $t \in [a, b]$.

    For any $x \in [a, b]$ and $n > N$,

$$\begin{aligned} |F_n(x) - F(x)| &= |\int_a^x (f'(t) - g(t))dt| \\ &\leq \int_a^x |f'(t) - g(t)|dt \\ &\leq \int_a^x \varepsilon dt \\ &\leq \varepsilon(b-a). \end{aligned}$$

    Hence, F_n converges uniformly to F.

    By fundamental theorem of calculus,

$$\int_a^x f_n'(t)dt = f_n(x) - f_n(a).$$

    Note that the left hand side will converge uniformly to $F(x)$ and the right hand side will converge uniformly to $f(x) - f(a)$. Hence, we obtain

$$\int_a^x g(t)dt = f(x) - f(a).$$

    Differentiating above identity, then we have $g(x) = f'(x)$.

12. One can check that e^{-nx^2} converges uniformly to 0 on $[1, 2]$, as $n \to \infty$: just note that $\|e^{-nx^2}\|_{[1,2]} = e^{-n} \to 0$ as $n \to \infty$. Hence

$$\lim_{n\to\infty} \int_1^2 e^{-nx^2} dx = \int_1^2 0dx = 0.$$

    Alternatively, when $x \in [1, 2]$, $e^{-nx^2} \leq e^{-n}$, thus

$$0 \leq \int_1^2 e^{-nx^2} dx \leq e^{-n}.$$

    Since $\lim_{n\to\infty} e^{-n} = 0$, hence

$$\lim_{n\to\infty} \int_1^2 e^{-nx^2} dx = 0.$$

13. When $a > 0$, $\frac{\sin nx}{nx}$ converges to 0 uniformly on $[a, \pi]$, since

$$\left\| \frac{\sin nx}{nx} \right\|_{[a,\pi]} \leq \frac{1}{na} \to 0$$

as $n \to \infty$. Hence
$$\lim_{n\to\infty}\int_a^\pi \frac{\sin nx}{nx}dx = \int_a^\pi 0dx = 0.$$
Now it is easy to check that $\frac{\sin x}{x}$ is a bounded function on $(0,\infty)$ (in fact $|\frac{\sin x}{x}| \le 1$ for all $x \in (0,\infty)$). Thus given $\varepsilon > 0$,
$$\left|\int_0^\varepsilon \frac{\sin nx}{nx}dx\right| \le \int_0^\varepsilon 1dx = \varepsilon$$
for all $n \in \mathbb{N}$. Since by our earlier result,
$$\lim_{n\to\infty}\int_\varepsilon^\pi \frac{\sin nx}{nx}dx = 0,$$
there exists $N \in \mathbb{N}$ such that for all $n \ge N$, we have
$$\left|\int_\varepsilon^\pi \frac{\sin nx}{nx}dx\right| < \varepsilon.$$
Together, whenever $n \ge N$, we have
$$\left|\int_0^\pi \frac{\sin nx}{nx}dx\right| < 2\varepsilon.$$
This proves
$$\lim_{n\to\infty}\int_0^\pi \frac{\sin nx}{nx}dx = 0.$$
Alternatively, we can proceed as follows: For $a > 0$,
$$|\int_a^\pi \frac{\sin(nx)}{nx}dx| \le \int_a^\pi \frac{|\sin(nx)|}{nx}dx \le \int_a^\pi \frac{1}{nx}dx = \frac{1}{n}(\frac{1}{a^2} - \frac{1}{\pi^2}).$$
Let $n \to \infty$, then
$$\lim_{n\to\infty}\int_a^\pi \frac{\sin(nx)}{nx}dx = 0.$$
For the case $a = 0$, changing of variable $y = nx$, then
$$\int_0^\pi \frac{\sin nx}{nx}dx = \frac{1}{n}\int_0^{n\pi}\frac{\sin y}{y}dy.$$
By Dirichlet test (or as we have seen in the last homework), we know that $\int_0^\infty \frac{\sin y}{y}dy$ converges, and we denote it by A. Therefore, $\lim_{n\to\infty}\int_0^{n\pi}\frac{\sin y}{y}dy = A$, and hence
$$\lim_{n\to\infty}\frac{1}{n}\int_0^{n\pi}\frac{\sin y}{y}dy = 0.$$

16. By definition, f_n is continuous except finite points $\{r_1, r_2 \dots r_n\}$, thus f_n is Riemann integrable. Obviously, $f_1(x) \le f_2(x) \dots f_n(x) \le \dots$ and $|f_n| \le 1$, hence $f(x) = \lim_{n\to\infty} f_n(x)$ is well defined.

    If x is irrational number, then $f_n(x) = 0$ for all n. So $f(x) = \lim_{n\to\infty} f_n(x) = 0$.

    If $x = r_k$ for some k, then $f_n(x) = 1$ for all $n \ge k$. So $f(x) = 1$.

    Hence, $f(x) = D(x)$, where $D(x)$ is Dirichlet function. D is not Riemann integrable on $[0,1]$.