MATH2060B Exercise 7

Deadline: Mar 10, 2015.

Supplementary Exercises

1. Show that the composition of two Riemann integrable functions may not be Riemann integrable. Hint: Take one to be the Thomae's function. (For comparison, let's remember the following result: if $g\colon [c, d] \to [a, b]$ is Riemann integrable, and $f\colon [a, b] \to \mathbb{R}$ is continuous, then $f \circ g$ is Riemann integrable on $[c, d]$.)

2. Find an antiderivative of the following functions. You should be as explicit as possible, giving piecewise defined functions with concrete formulas whenever appropriate.

   (a) $|x|$

   (b) $x|x|$

   (c) $|\sin x|$

3. Evaluate the following limits:

   (a) $\displaystyle\lim_{n\to\infty} \left(\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{n+n}\right)$

   (b) $\displaystyle\lim_{n\to\infty} \frac{(n!)^{1/n}}{n}$

4. Evaluate the following integral
$$\int_0^\pi \frac{x \sin x}{1+\cos^2 x} dx.$$

5. Explain why each of the following integrals must be interpreted as an improper integral. Then determine whether each of these improper integrals is convergent. Also find the values of those improper integrals that are convergent.

   (a) $\displaystyle\int_0^\infty e^{-x} dx$

   (b) $\displaystyle\int_0^\infty xe^{-x} dx$

   (c) $\displaystyle\int_1^\infty \frac{1}{\sqrt{x}} dx$

   (d) $\displaystyle\int_0^1 \frac{1}{\sqrt{x}} dx$

   (e) $\displaystyle\int_0^1 \frac{1}{x} dx$

   (f) $\displaystyle\int_0^{1/e} \frac{1}{x(\ln x)^2} dx$

   (g) $\displaystyle\int_0^1 \frac{1}{\sqrt{1-x^2}} dx$

6. Prove the following form of the absolute convergence test for improper integrals: Suppose $f\colon (a,b] \to \mathbb{R}$ is a function that is Riemann integrable on $[c,b]$ for all $c \in (a,b]$. If the improper integral $\int_a^b |f|$ exists, then the improper integral $\int_a^b f$ exists.

7. Prove the following form of comparison test for improper integrals: Suppose $f, g\colon (a,b] \to \mathbb{R}$ are Riemann integrable on $[c,b]$ for all $c \in (a,b]$. Suppose also $0 \le f(x) \le g(x)$ for all $x \in (a,b]$. Then the improper integral $\int_a^b f$ converges, if the improper integral $\int_a^b g$ converges.

8. Determine whether each of the following improper integrals is convergent.

   (a) $\displaystyle\int_1^\infty \frac{x+1}{x^3+x^2}dx$

   (b) $\displaystyle\int_1^\infty \frac{\ln x}{x^2}dx$

   (c) $\displaystyle\int_0^{1/e} \frac{e^x}{x|\ln x|}dx$

   (d) $\displaystyle\int_1^\infty \frac{\sin x}{x\sqrt{1+x^2}}dx$

   (e) $\displaystyle\int_1^\infty \frac{\sin x}{x}dx$