Solution 5

Section 7.1

1 (a) $\|\mathcal{P}_1\| = 2.$

2 (a) $\mathcal{S}(x^2, \mathcal{P}_1) = 9.$

(b) $\mathcal{S}(x^2, \mathcal{P}_1) = 37.$

14. (a) $\dfrac{1}{3}(x_{i-1}^2 + x_{i-1}x_{i-1} + x_{i-1}^2) \leq q_i^2 := \dfrac{1}{3}(x_i^2 + x_i x_{i-1} + x_{i-1}^2) \leq \dfrac{1}{3}(x_i^2 + x_i x_i + x_i^2)$
$\Rightarrow \quad 0 \leq x_{i-1}^2 \leq q_i^2 \leq x_i^2 \quad \Rightarrow \quad 0 \leq x_{i-1} \leq q_i \leq x_i.$

(b) $Q(q_i)(x_i - x_{i-1}) = \dfrac{1}{3}(x_i^2 + x_i x_{i-1} + x_{i-1}^2)(x_i - x_{i-1}) = \dfrac{1}{3}(x_i^3 - x_{i-1}^3).$

(c) $S(Q; \dot{\mathcal{Q}}) = \displaystyle\sum_{i=1}^{n} Q(q_i)(x_i - x_{i-1}) = \frac{1}{3}\sum_{i=1}^{n}(x_i^3 - x_{i-1}^3) = \frac{1}{3}(b^3 - a^3).$

(d) Let $\varepsilon > 0$, and $\dot{\mathcal{P}} := \{([x_{i-1}, x_i], t_i)\}_{i=1}^n$ be a tagged partition of $[a, b]$ with $\| \dot{\mathcal{P}} \| < \delta$. By (a) and since $\dot{\mathcal{Q}}$ has the same partition points, $q_i, t_i \in [x_{i-1}, x_i] \;\Rightarrow\; |t_i - q_i| < \delta$.

$$
\begin{aligned}
&| \; S(Q; \dot{\mathcal{P}}) - S(Q; \dot{\mathcal{Q}}) \; | \\
&= \left| \sum_{i=1}^{n} Q(t_i)\Delta x_i - \sum_{i=1}^{n} Q(q_i)\Delta x_i \right| = \left| \sum_{i=1}^{n} (t_i^2 - q_i^2)\Delta x_i \right| \\
&\leq \sum_{i=1}^{n} | \; t_i^2 - q_i^2 \; |\Delta x_i = \sum_{i=1}^{n} | \; t_i - q_i \; | \; | \; t_i + q_i \; |\Delta x_i \\
&< \delta \sum_{i=1}^{n} (| \; t_i \; | + | \; q_i \; |)\Delta x_i \leq 2b\delta \sum_{i=1}^{n} \Delta x_i = 2b(b-a)\delta.
\end{aligned}
$$

Choose $\delta := \varepsilon/(2b(b-a))$. Then $| \; S(Q; \dot{\mathcal{P}}) - S(Q; \dot{\mathcal{Q}}) \; | < \varepsilon$.

Hence $Q \in \mathcal{R}[a, b]$ and $\displaystyle\int_a^b Q = \int_a^b x^2 dx = S(Q; \dot{\mathcal{Q}}) = \frac{1}{3}(b^3 - a^3).$

Section 7.2

2. Let $\mathcal{P}_n : x_0 = 0 < x_1 = \frac{1}{n} < x_2 = \frac{2}{n} < \cdots < x_n = 1$ be partition of [0,1] and tag $t_i = x_i$.

   Then the Riemann sum $\mathcal{S}(h, \mathcal{P}_n) = \sum_{i=1}^{n} \frac{1}{n}(\frac{i}{n} + 1) = \frac{n+1}{2n} + 1$.

   Let $n \to \infty$, then $\lim_{n\to\infty} \mathcal{S}(h, \mathcal{P}_n) = \frac{3}{2}$.

   On the other hand, let $\mathcal{Q}_n : x_0 = 0 < x_1 = \frac{1}{n} < x_2 = \frac{2}{n} < \cdots < x_n = 1$ be partition of [0,1] and tag t_i be irrational number $\in (x_{i-1}, x_{i+1})$.

   Then the Riemann sum $\mathcal{S}(h, \mathcal{Q}_n) = 0$, thus $\lim_{n\to\infty} \mathcal{S}(h, \mathcal{Q}_n) = 0$.

   Hence, $h \notin \mathcal{R}[0, 1]$.

12. Given $\varepsilon > 0$, let h_ε be the restriction of $g(x) = \sin 1/x$ on $[\varepsilon, 1]$. Then since h_ε is continuous on $[\varepsilon, 1]$ (which in particular implies h_ε is Riemann integrable on $[\varepsilon, 1]$), there exists a partition Q of $[\varepsilon, 1]$ such that

$$\overline{S}(h_\varepsilon, Q) - \underline{S}(h_\varepsilon, Q) < \varepsilon.$$

   Now let P be the partition of $[0, 1]$ defined by $P = \{0\} \cup Q$. Then

$$\overline{S}(g, P) - \underline{S}(g, P) = \left(\operatorname{osc}_{[0,\varepsilon]} g\right) \varepsilon + \overline{S}(h_\varepsilon, Q) - \underline{S}(h_\varepsilon, Q) < 1 \cdot \varepsilon + \varepsilon = 2\varepsilon.$$

   Hence g is Riemann integrable on $[0, 1]$.

Section 7.4

12. Let $f(x) = x^2$ for $x \in [0, 1]$, and $P_n = \{0, 1/n, 2/n, \ldots, (n-1)/n, 1\}$. Then

$$L(f, P_n) = \sum_{i=1}^{n} \left(\frac{i-1}{n}\right)^2 \frac{1}{n} = \frac{1^2 + 2^2 + \cdots + (n-1)^2}{n^3} = \frac{n(n-1)(2n-1)}{6n^3} = \frac{(n-1)(2n-1)}{6n^2}.$$

$$U(f, P_n) = \sum_{i=1}^{n} \left(\frac{i}{n}\right)^2 \frac{1}{n} = \frac{1^2 + 2^2 + \cdots + n^2}{n^3} = \frac{n(n+1)(2n+1)}{6n^3} = \frac{(n+1)(2n+1)}{6n^2}.$$

   Hence

$$L(f) \geq \sup_n L(f, P_n) = \frac{1}{3},$$

   and

$$U(f) \leq \inf_n U(f, P_n) = \frac{1}{3}.$$

   Since $U(f) \geq L(f)$, this forces $L(f) = U(f) = \frac{1}{3}$.

Supplementary Exercises

1. (a)
$$\begin{aligned}
\overline{S}(f,\mathcal{P}) &= \sum_{j=1}^{4} \sup_{I_j} f \ \Delta x_j \\
&= \left(\sup_{x\in[-1,-1/2]} -x\right)\left(-\frac{1}{2}-(-1)\right) + \left(\sup_{x\in[-1/2,0]} -x\right)\left(0-\left(-\frac{1}{2}\right)\right) \\
&\qquad + \left(\sup_{x\in[0,1/3]} -x+1\right)\left(\frac{1}{3}-0\right) + \left(\sup_{x\in[1/3,1]} -x+1\right)\left(1-\frac{1}{3}\right) \\
&= (1)\left(-\frac{1}{2}-(-1)\right) + \left(\frac{1}{2}\right)\left(0-\left(-\frac{1}{2}\right)\right) + (1)\left(\frac{1}{3}-0\right) + \left(\frac{2}{3}\right)\left(1-\frac{1}{3}\right) \\
&= \frac{55}{36} \\
\underline{S}(f,\mathcal{P}) &= \sum_{j=1}^{4} \inf_{I_j} f \ \Delta x_j \\
&= \left(\inf_{x\in[-1,-1/2]} -x\right)\left(-\frac{1}{2}-(-1)\right) + \left(\inf_{x\in[-1/2,0]} -x\right)\left(0-\left(-\frac{1}{2}\right)\right) \\
&\qquad + \left(\inf_{x\in[0,1/3]} -x+1\right)\left(\frac{1}{3}-0\right) + \left(\inf_{x\in[1/3,1]} -x+1\right)\left(1-\frac{1}{3}\right) \\
&= \left(\frac{1}{2}\right)\left(-\frac{1}{2}-(-1)\right) + 0\left(0-\left(-\frac{1}{2}\right)\right) + \left(\frac{2}{3}\right)\left(\frac{1}{3}-0\right) + 0\left(1-\frac{1}{3}\right) \\
&= \frac{17}{36}
\end{aligned}$$

The Darboux upper sum is not a Riemann sum because $\sup\limits_{[0,1/3]} f = 1$ but we can't find any tag $z \in [0, 1/3]$ so that $f(z) = 1$, because of the definition of f.

(b) Take $\mathcal{P}_n := \left\{ x_i := -1 + \dfrac{i}{n} \right\}_{i=0}^{2n}$, hence $\| \mathcal{P}_n \| \to 0$.

Then
$$\begin{aligned}
\overline{S}(f) = \lim \overline{S}(f,\mathcal{P}_n) &= \lim \left(\sum_{i=1}^{n} (-x_{i-1}) \ \Delta x_i + \sum_{i=n+1}^{2n} (-x_{i-1}+1) \ \Delta x_i \right) \\
&= \lim \left(\sum_{i=1}^{2n} (-x_{i-1}) \ \Delta x_i + \sum_{i=n+1}^{2n} \Delta x_i \right) \\
&= \lim \left(\sum_{i=1}^{2n} \left(1-\frac{i-1}{n}\right)\left(\frac{1}{n}\right) + \sum_{i=n+1}^{2n} \left(\frac{1}{n}\right) \right) = 2 - \lim \frac{1}{n^2} \sum_{i=1}^{2n} (i-1) + 1 \\
&= 3 - \lim \frac{1}{n^2} \frac{(0+(2n-1))2n}{2} = 3 - \lim \frac{2n-1}{n} = 3 - 2 = 1
\end{aligned}$$

$$\text{and} \quad \underline{S}(f) = \lim \underline{S}(f, \mathcal{P}_n) = \lim \left(\sum_{i=1}^{n} (-x_i)\ \Delta x_i + \sum_{i=n+1}^{2n} (-x_i + 1)\ \Delta x_i \right)$$
$$= \lim \left(\sum_{i=1}^{2n} (-x_i)\ \Delta x_i + \sum_{i=n+1}^{2n} \Delta x_i \right)$$
$$= \lim \left(\sum_{i=1}^{2n} \left(1 - \frac{i}{n}\right) \left(\frac{1}{n}\right) + \sum_{i=n+2}^{2n} \left(\frac{1}{n}\right) \right) = 2 - \lim \frac{1}{n^2} \sum_{i=1}^{2n} i + 1$$
$$= 3 - \lim \frac{1}{n^2} \frac{(1+2n)2n}{2} = 3 - \lim \frac{1+2n}{n} = 3 - 2 = 1$$

Hence $\overline{S}(f) = 1 = \underline{S}(f)$, by integrability criterion, $f \in \mathcal{R}[-1,1]$ and $\int_{-1}^{1} f = 1$

2. Since f is Riemann integrable on $[a,b]$, given $\varepsilon > 0$, there exist a partition P_0 of $[a,b]$ such that
$$\overline{S}(f, P_0) - \underline{S}(f, P_0) < \varepsilon.$$
Let P_1 be a refinement of P_0 with $P_1 = P_0 \cup \{c, d\}$. Then
$$\overline{S}(f, P_1) - \underline{S}(f, P_1) \leq \overline{S}(f, P_0) - \underline{S}(f, P_0) < \varepsilon.$$
Let P be a partition of $[c,d]$ given by $P = P_1 \cap [c,d]$. Then
$$\overline{S}(f, P) - \underline{S}(f, P) \leq \overline{S}(f, P_1) - \underline{S}(f, P_1) < \varepsilon;$$
here the first inequality follows, since if $P_1 = \{a = x_0 < \cdots < x_{i_0} = c < \cdots < x_{i_1} = d < \ldots x_n = b\}$, then
$$\begin{aligned}\overline{S}(f, P) - \underline{S}(f, P) &= \sum_{i=1}^{n} \left(\operatorname{osc}_{[x_{i-1}, x_i]} f\right)(x_i - x_{i-1}) \\ &\geq \sum_{i=i_0+1}^{i_1} \left(\operatorname{osc}_{[x_{i-1}, x_i]} f\right)(x_i - x_{i-1}) \\ &= \overline{S}(f, P_1) - \underline{S}(f, P_1).\end{aligned}$$
Since this is true for any $\varepsilon > 0$, we have $f \in \mathcal{R}[c,d]$.

3. Since f is integrable, given $\varepsilon > 0$, then there exists partition $\mathcal{P} = \{I_j\}$ of $[a,b]$ such that
$$\sum_j osc_{I_j} f < \varepsilon.$$
By triangle inequality, $osc_{I_j}|f| \leq osc_{I_j} f$, then
$$\sum_j osc_{I_j}|f| \leq \sum_j osc_{I_j} f < \varepsilon.$$
Hence $|f|$ is Riemann integrable.

The converse is not true. For example, taking function f to be 1 when x is rational number in $[0,1]$ and be -1 when x is irrational number in $[0,1]$. Then $|f|(x) = 1$ for each point, so $|f|$ is integrable. However, f is not integrable for the following reason:

Let $\mathcal{P}_n : x_0 = 0 < x_1 = \frac{1}{n} < x_2 = \frac{2}{n} < \cdots < x_n = 1$ be partition of [0,1] and tag $t_i = x_i$. Then the Riemann sum $\mathcal{S}(f, \mathcal{P}_n) = 1$.

However, let $\mathcal{Q}_n : x_0 = 0 < x_1 = \frac{1}{n} < x_2 = \frac{2}{n} < \cdots < x_n = 1$ be partition of [0,1] and tag t_i be irrational number . Then the Riemann sum $\mathcal{S}(f, \mathcal{Q}_n) = -1$.

Hence, f is not integrable.

4. (a) Since f,g are Riemann integrable, given $\varepsilon > 0$, there exists $\delta > 0$ such that for any tagged partition $\mathcal{P}$ with $\|\mathcal{P}\| \leq \delta$, then

$$|\int_a^b f - \mathcal{S}(f, \mathcal{P})| < \varepsilon,$$

$$|\int_a^b g - \mathcal{S}(g, \mathcal{P})| < \varepsilon.$$

Since $f \leq g$, thus $\mathcal{S}(f, \mathcal{P}) \leq \mathcal{S}(g, \mathcal{P})$. Therefore,

$$\int_a^b f \leq \mathcal{S}(f, \mathcal{P}) + \varepsilon \leq \mathcal{S}(g, \mathcal{P}) + \varepsilon \leq \int_a^b g + 2\varepsilon.$$

Let $\varepsilon \to 0^+$, then $\int_a^b f \leq \int_a^b g$.

(b) Since $-|f| \leq f \leq |f|$, by (a), we have $-\int_a^b |f| \leq \int_a^b f \leq \int_a^b |f|$. Hence,

$$|\int_a^b f| \leq \int_a^b |f|.$$

If $|f| \leq M$,

$$\int_a^b |f| \leq \int_a^b M = M(b-a).$$

5. It suffices to show that $\frac{1}{g}$ is Riemann integrable. Since g is integrable, given $\varepsilon > 0$, then there exists partition $\mathcal{P} = I_j$ of $[a, b]$ such that

$$\sum_j osc_{I_j} g < \varepsilon.$$

Since $|g| > c$, thus $osc_{I_j} \frac{1}{g} < \frac{1}{c^2} osc_{I_j} g < \frac{\varepsilon}{c^2}$. Hence, $\frac{1}{g}$ is Riemann integrable.

6. Since $\inf f + \inf g \leq \inf(f+g)$, thus the Darboux sum $\underline{S}(f, P) + \underline{S}(g, P) \leq \underline{S}(f+g, P)$ for any partition P. For any $\varepsilon > 0$, by definition of supermen, there exists partition P_1 and P_2 such that

$$\underline{S}(f) - \varepsilon \leq \underline{S}(f, P_1),$$
$$\underline{S}(g) - \varepsilon \leq \underline{S}(g, P_2).$$

Let P be the refinement of P_1 and P_2, by Proposition 2.1 in lecture note, then

$$\underline{S}(f) - \varepsilon \leq \underline{S}(f, P),$$

$$\underline{S}(g) - \varepsilon \leq \underline{S}(g, P).$$

Therefore, we have

$$\underline{S}(f) + \underline{S}(g) - 2\varepsilon \leq \underline{S}(f, P) + \underline{S}(g, P) \leq \underline{S}(f+g, P) \leq \underline{S}(f+g).$$

Let $\varepsilon \to 0^+$, we obtain the result.

Taking function f to be 1 when x is rational number in $[0, 1]$ and be -1 when x is irrational number in $[0, 1]$ and $g = -f$. So $\underline{S}(f + g) = \underline{S}(0) = 0$. By the calculation in exercise 3, we know that $\underline{S}(f) = -1$. Using the same argument we can get $\underline{S}(g) = -1$. Hence,

$$0 = \underline{S}(f + g) > \underline{S}(f) + \underline{S}(g) = -2.$$