MATH2060B Exercise 4

Deadline: Feb 3, 2015.

The questions are from Bartle and Sherbert, *Introduction to Real Analysis*, Wiley, 4th edition, unless otherwise stated.

Section 6.4 Q.3, 10, 12, 14(b)(d).

Supplementary Exercises

1. Let f be a continuous function on (a, b) satisfying
$$f\left(\frac{x+y}{2}\right) \le \frac{1}{2}\Big(f(x)+f(y)\Big), \quad \forall x, y \in (a, b).$$
Show that f is convex.

2. Let f be differentiable on (a, b). Show that it is convex if and only if
$$f(y) - f(x) \ge f'(x)(y-x), \qquad \forall x, y \in (a, b).$$

3. Give an example to show that the product of two convex functions may not be convex. How about the composition of two convex functions?

4. (a) For any convex function f on (a, b), prove Jensen's inequality
$$f(\lambda_1 x_1 + \cdots + \lambda_n x_n) \le \lambda_1 f(x_1) + \cdots + \lambda_n f(x_n),$$
for $x_1, \cdots, x_n \in (a, b)$, $\lambda_1, \cdots, \lambda_n \in (0, 1)$, $\sum_j \lambda_j = 1$.
   (b) Show that strict inequality in Jensen's inequality holds when f is strictly convex and not all x_j are equal.
   (c) Prove the following inequality about arithmetic and geometric means:
$$(y_1 \ldots y_n)^{1/n} \le \frac{y_1 + \cdots + y_n}{n}$$
for any positive real numbers $y_1, \ldots, y_n$, and equality holds if and only if $y_1 = \cdots = y_n$. (Hint: Let $y_i = e^{x_i}$, and use (a) and (b) for an appropriate choice of f and λ_i's.)

5. Prove Young's inequality: if $1 < p < \infty$ and q is the conjugate exponent of p (in the sense that $\frac{1}{p} + \frac{1}{q} = 1$), then
$$xy \le \frac{x^p}{p} + \frac{y^q}{q}, \quad \forall x, y > 0.$$
Moreover, equality in this inequality holds if and only if $x^p = y^q$. (Hint: write $x^p = e^{t_1}$ and $y^q = e^{t_2}$, and apply convexity.)

6. Prove Hölder's inequality: if $1 < p < \infty$ and q is the conjugate exponent of p (in the sense that $\frac{1}{p} + \frac{1}{q} = 1$), then for any positive real numbers $a_1, \ldots, a_k, b_1, \ldots, b_k$, we have
$$\sum_{k=1}^{n} a_k b_k \le \Big(\sum_{k=1}^{n} a_k^p\Big)^{1/p} \Big(\sum_{k=1}^{n} b_k^q\Big)^{1/q}.$$

Also characterize its equality case. (Hint: Without loss of generality we may assume, in addition, that

$$\sum_{k=1}^{n} a_k^p = \sum_{k=1}^{n} b_k^q = 1.$$

(Why?) Once we have this reduction, we apply Young's inequality as in Question 5, with $x = a_k$, $y = b_k$, and sum the result in k. This will also help characterize the equality case.)