Solution 3

Section 6.3

4. It is clear that both $f(0)$ and $g(0)$ are zero. Since

$$\left|\frac{f(x)-f(0)}{x-0}\right| = \frac{f(x)}{|x|} \leq |x|$$

for all $x \neq 0$, by Sandwich theorem, $f'(0)$ exists and equals 0. Also, it is easy to see that $g'(0)$ exists, and equals $\cos 0 = 1 \neq 0$. Thus Theorem 6.3.1 applies, and

$$\lim_{x\to 0}\frac{f(x)}{g(x)} = \frac{f'(0)}{g'(0)} = 0.$$

We cannot apply L'Hopital's rule Theorem 6.3.1, because $f'(x)$ fails to exist in any deleted neighborhood of 0. In fact, $f'(x)$ does not exist at any point x other than 0, since f is not even continuous at any point other than 0.

10. (d) Note that for $x \neq 0$,

$$\frac{1}{x} - \frac{1}{\arctan x} = \frac{\arctan x - x}{x \arctan x}.$$

We apply L'Hopital's rule to compute its limit as $x \to 0$. First, we check that both

$$\lim_{x\to 0}(\arctan x - x) = 0, \quad \text{and} \quad \lim_{x\to 0} x \arctan x = 0.$$

Next, both $\arctan x - x$ and $x \arctan x$ are differentiable on a deleted neighborhood of 0, and their derivatives are

$$\frac{d}{dx}(\arctan x - x) = \frac{1}{1+x^2} - 1,$$

$$\frac{d}{dx}(x \arctan x) = \frac{x}{1+x^2} + \arctan x.$$

We claim that

$$\lim_{x\to 0}\frac{\frac{1}{1+x^2}-1}{\frac{x}{1+x^2}+\arctan x} \quad \text{exists and equals 0;}$$

then by L'Hopital's rule, the given limit

$$\lim_{x\to 0}\left(\frac{1}{x} - \frac{1}{\arctan x}\right) \quad \text{also exists and equals 0.}$$

To check the claim, we use L'Hopital's rule again: we check that

$$\lim_{x\to 0}\left(\frac{1}{1+x^2} - 1\right) = 0, \quad \text{and} \quad \lim_{x\to 0}\left(\frac{x}{1+x^2} + \arctan x\right) = 0.$$

Also, both the numerator and denominator are differentiable on a deleted neighborhood of 0, and their derivatives are

$$\frac{d}{dx}\left(\frac{1}{1+x^2} - 1\right) = -\frac{2x}{(1+x^2)^2},$$

$$\frac{d}{dx}\left(\frac{x}{1+x^2}+\arctan x\right)=\frac{2}{1+x^2}-\frac{2x^2}{(1+x^2)^2}=\frac{2}{(1+x^2)^2}.$$

Thus

$$\frac{\frac{d}{dx}\left(\frac{1}{1+x^2}-1\right)}{\frac{d}{dx}\left(\frac{x}{1+x^2}+\arctan x\right)}=-x,$$

and it converges to 0 as $x \to 0$. Hence by L'Hopital's rule,

$$\lim_{x\to 0}\frac{\frac{1}{1+x^2}-1}{\frac{x}{1+x^2}+\arctan x} \quad \text{exists and equals } 0,$$

as we have claimed.

11. (c) Note that $x^{\sin x}=e^{\sin x \ln x}$ for $x>0$. We claim that $\lim_{x\to 0^+}\sin x \ln x$ exists and equals 0. Then by continuity of exp at 0, we see that

$$\lim_{x\to 0^+} x^{\sin x} \quad \text{exists, and is equal to } e^0=1.$$

Now we prove the claim. To do so, note that for $x>0$,

$$\sin x \ln x = \frac{\ln x}{\csc x}.$$

We compute its limit as $x\to 0^+$ using L'Hopital's rule: first we check

$$\lim_{x\to 0^+}\ln x=-\infty, \quad \lim_{x\to 0^+}\csc x=+\infty.$$

(Technically we only need the limit of the denominator here, but it's a good habit to also check the limit of the numerator to see that it is ∞/∞, because otherwise one usually does not need to evaluate that limit by L'Hopital's rule.) Also, both $\ln x$ and $\csc x$ are differentiable on $(0,\pi)$, with

$$\frac{d}{dx}\ln x=\frac{1}{x},$$

$$\frac{d}{dx}\csc x=-\cot^2 x.$$

So our claim will follow from L'Hopital's rule, if we can show

$$\lim_{x\to 0^+}\frac{\frac{1}{x}}{-\cot^2 x} \quad \text{exists and equals } 0.$$

But the latter is easy to see:

$$\frac{\frac{1}{x}}{-\cot^2 x}=-\frac{\sin^2 x}{x\cos^2 x}=-\frac{\sin x}{x}\frac{\sin x}{\cos^2 x}$$

for $x\neq 0$. As $x\to 0^+$,

$$\frac{\sin x}{x}\to 1, \quad \frac{\sin x}{\cos^2 x}\to 0,$$

so

$$\lim_{x\to 0^+}\frac{\frac{1}{x}}{-\cot^2 x} \quad \text{exists, and equals } 1\cdot 0=0.$$

12. Note $f(x) = \dfrac{e^x f(x)}{e^x}$, and since f and $x \mapsto e^x$ is differentiable on $(0, +\infty)$, $x \mapsto e^x f(x)$ is differentiable on $(0, +\infty)$. Moreover, $\lim\limits_{x\to+\infty} e^x = +\infty$. By L'Hôpital rule II (Theorem 6.3.5), we have
$$\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty} \frac{e^x f(x)}{e^x} = \lim_{x\to+\infty} \frac{e^x f(x) + e^x f'(x)}{e^x} = \lim_{x\to+\infty} (f(x) + f'(x)) = L.$$
Hence, $\lim\limits_{x\to+\infty} f'(x) = \lim\limits_{x\to+\infty} (f(x) + f'(x)) - \lim\limits_{x\to+\infty} f(x) = L - L = 0.$

Section 6.4

16. We compute $\lim\limits_{h\to 0} \dfrac{f(a+h) - 2f(a) + f(a-h)}{h^2}$ using L'Hopital's rule.

First, we check that
$$\lim_{h\to 0} f(a+h) - 2f(a) + f(a-h) = 0 \quad \text{and} \quad \lim_{h\to 0} h^2 = 0.$$
(The first limit follows from continuity of f at a, which follows since f is differentiable at a.) Next, we note that both $f(a+h) - 2f(a) + f(a-h)$ and h^2 are differentiable functions of h, with
$$\frac{d}{dh}(f(a+h) - 2f(a) + f(a-h)) = f'(a+h) - f'(a-h)$$
and
$$\frac{d}{dh} h^2 = 2h.$$
Since
$$\frac{f'(a+h) - f'(a-h)}{2h} = \frac{f'(a+h) - f'(a)}{2h} + \frac{f'(a) - f'(a-h)}{2h},$$
which converges to $\frac{f''(a)}{2} + \frac{f''(a)}{2} = f''(a)$ as $h \to 0$ (the limit here follows since $f''(a)$ exist), we then have, from L'Hopital's rule, that
$$\lim_{h\to 0} \frac{f(a+h) - 2f(a) + f(a-h)}{h^2} \quad \text{exists, and equals } f''(a).$$

Observation After putting $a = 0$, any odd function makes the limit exist and equal to 0.

e.g. Take $f(x) := \operatorname{sgn} x$. Hence $f''(0)$ doesn't exist. But
$$\lim_{h\to 0} \frac{f(0+h) - 2f(0) + f(0-h)}{h^2} = \lim_{h\to 0} \frac{1 - 2(0) - 1}{h^2} = 0.$$

18. By Taylor theorem, $\exists\ \xi,\ \eta$ between x and c s.t.
$$f(x) = f(c) + f'(c)(x-c) + \cdots + \frac{f^{(n)}(\xi)}{n!}(x-c)^n = \frac{f^{(n)}(\xi)}{n!}(x-c)^n$$
$$g(x) = g(c) + g'(c)(x-c) + \cdots + \frac{g^{(n)}(\eta)}{n!}(x-c)^n = \frac{g^{(n)}(\eta)}{n!}(x-c)^n$$
Since $f^{(n)}$ is continuous on I, $\lim\limits_{x\to c} f^{(n)}(\xi) = \lim\limits_{\xi\to c} f^{(n)}(\xi) = f^{(n)}(c)$.

Similarly, $\lim\limits_{x\to c} g^{(n)}(\eta) = g^{(n)}(c)$. Hence, $\lim\limits_{x\to c} \dfrac{f(x)}{g(x)} = \lim\limits_{x\to c} \dfrac{f^{(n)}(\xi)}{g^{(n)}(\eta)} = \dfrac{f^{(n)}(c)}{g^{(n)}(c)}$.

Supplementary Exercise

1. Define $g(x) = \dfrac{f(x)}{x}$, then g is differentiable on $(0, \infty)$. Since f is continuous on $[0, \infty)$, differentiable on $(0, \infty)$ with $f(0) = 0$, we have, by Mean Value Theorem, $\exists\ \xi \in (0, \infty)$ such that
$$\frac{f(x)}{x} = \frac{f(x) - f(0)}{x - 0} = f'(\xi) < f'(x).$$
If $x > 0$
$$g'(x) = \frac{xf'(x) - f(x)}{x^2} = \frac{f'(x)}{x} - \frac{f(x)}{x^2} = \frac{1}{x}\left(f'(x) - \frac{f(x)}{x}\right) > 0$$
Hence g is strictly increasing on $(0, \infty)$, i.e. $\dfrac{f(x)}{x} < \dfrac{f(y)}{y}$ if $0 < x < y$.

2. Define
$$H(x) := \det\begin{pmatrix} f(a) & g(a) & h(a) \\ f(b) & g(b) & h(b) \\ f(x) & g(x) & h(x) \end{pmatrix}$$
Since f, g, h is continuous in $[a, b]$ and differentiable in (a, b), then $H(x)$ is continuous in $[a, b]$ and differentiable in (a, b). Obviously, $H(a) = H(b) = 0$. By Rolle's theorem, $\exists\ c \in (a, b)$ s.t. $H'(c) = 0$.

Method 1 − Differentiation Formula of Determinant
$$\begin{aligned} H'(x) &= \det\begin{pmatrix} (f(a))' & (g(a))' & (h(a))' \\ f(b) & g(b) & h(b) \\ f(x) & g(x) & h(x) \end{pmatrix} + \det\begin{pmatrix} f(a) & g(a) & h(a) \\ (f(b))' & (g(b))' & (h(b))' \\ f(x) & g(x) & h(x) \end{pmatrix} \\ &\quad + \det\begin{pmatrix} f(a) & g(a) & h(a) \\ f(b) & g(b) & h(b) \\ f'(x) & g'(x) & h'(x) \end{pmatrix} \\ &= \det\begin{pmatrix} 0 & 0 & 0 \\ f(b) & g(b) & h(b) \\ f(x) & g(x) & h(x) \end{pmatrix} + \det\begin{pmatrix} f(a) & g(a) & h(a) \\ 0 & 0 & 0 \\ f(x) & g(x) & h(x) \end{pmatrix} + \det\begin{pmatrix} f(a) & g(a) & h(a) \\ f(b) & g(b) & h(b) \\ f'(x) & g'(x) & h'(x) \end{pmatrix} \\ &= \det\begin{pmatrix} f(a) & g(a) & h(a) \\ f(b) & g(b) & h(b) \\ f'(x) & g'(x) & h'(x) \end{pmatrix} \end{aligned}$$

Method 2 − Direct Observation
$$H(x) = \det\begin{pmatrix} g(a) & h(a) \\ g(b) & h(b) \end{pmatrix} f(x) - \det\begin{pmatrix} f(a) & h(a) \\ f(b) & h(b) \end{pmatrix} g(x) + \det\begin{pmatrix} f(a) & g(a) \\ f(b) & g(b) \end{pmatrix} h(x)$$
$$\begin{aligned} H'(x) &= \det\begin{pmatrix} g(a) & h(a) \\ g(b) & h(b) \end{pmatrix} f'(x) - \det\begin{pmatrix} f(a) & h(a) \\ f(b) & h(b) \end{pmatrix} g'(x) + \det\begin{pmatrix} f(a) & g(a) \\ f(b) & g(b) \end{pmatrix} h'(x) \\ &= \det\begin{pmatrix} f(a) & g(a) & h(a) \\ f(b) & g(b) & h(b) \\ f'(x) & g'(x) & h'(x) \end{pmatrix} \end{aligned}$$
We conclude that
$$\det\begin{pmatrix} f(a) & g(a) & h(a) \\ f(b) & g(b) & h(b) \\ f'(c) & g'(c) & h'(c) \end{pmatrix} = H'(c) = 0.$$

3. The solution is wrong, since in applying the L'Hopital's rule, one has to first show that

$$\lim_{x\to a}\frac{f'(x)}{g'(x)}$$

exists. If this limit does not exist, then one can conclude nothing about

$$\lim_{x\to a}\frac{f(x)}{g(x)}$$

from the L'Hopital's rule.

The correct solution is as follows: Note that for $x \neq 0$,

$$\frac{x^2\sin\frac{1}{x}}{\sin x} = \frac{x}{\sin x}\cdot x\sin\frac{1}{x},$$

Also,

$$\lim_{x\to 0}\frac{x}{\sin x} \quad \text{exists, and equals 1,}$$

$$\lim_{x\to 0} x\sin\frac{1}{x} \quad \text{exists, and equals 0}$$

(this last limit follows from Sandwich theorem, since $-|x| \leq x\sin\frac{1}{x} \leq |x|$ for all $x \neq 0$, and $\lim_{x\to 0} -|x| = \lim_{x\to 0}|x| = 0$). So we get

$$\lim_{x\to 0}\frac{x^2\sin\frac{1}{x}}{\sin x} \quad \text{exists, and equals } 1\cdot 0 = 0.$$