## MATH2060B Exercise 3

**Deadline:** Jan 27, 2015.

The questions are from Bartle and Sherbert, *Introduction to Real Analysis*, Wiley, 4th edition, unless otherwise stated.

**Section 6.3** Q.4, 10(d), 11(c), 12;
**Section 6.4** Q.16, 18.

### Supplementary Exercises

1. Let $f : [0,\infty) \to \mathbb{R}$ be continuous on $[0,\infty)$, and differentiable on $(0,\infty)$. Suppose $f(0) = 0$, and $f'$ is strictly increasing on $(0,\infty)$. Prove that if $0 < x < y$, then
$$\frac{f(x)}{x} < \frac{f(y)}{y}.$$

2. Suppose $f, g, h$ are continuous on $[a,b]$ and differentiable on $(a,b)$. Show that $\exists\, c \in (a,b)$ such that
$$\det \begin{pmatrix} f(a) & g(a) & h(a) \\ f(b) & g(b) & h(b) \\ f'(c) & g'(c) & h'(c) \end{pmatrix} = 0.$$

   Remark: When $h$ is identically equal to 1, the above is simply Cauchy's mean value theorem. (You are not required to prove this last observation.)

3. A student was asked to compute the following limit:
$$\lim_{x\to 0} \frac{x^2 \sin\frac{1}{x}}{\sin x}.$$

   Here is her answer script:

   *Both* $x^2 \sin\frac{1}{x}$ *and* $\sin x$ *are differentiable in a deleted neighborhood of* $0$*, and* $\frac{d}{dx}\sin x = \cos x$ *which is non-zero in a deleted neighborhood of* $0$*. Thus L'Hopital's rule applies, and*
$$\lim_{x\to 0} \frac{x^2 \sin\frac{1}{x}}{\sin x} = \lim_{x\to 0} \frac{\frac{d}{dx}\left(x^2 \sin\frac{1}{x}\right)}{\frac{d}{dx}\sin x}.$$
   *But the last limit is*
$$\lim_{x\to 0} \frac{2x \sin\frac{1}{x} - \cos\frac{1}{x}}{\cos x},$$
   *which does not exist; this is because* $\lim\limits_{x\to 0} \cos x = 1$, $\lim\limits_{x\to 0} 2x \sin\frac{1}{x} = 0$ *(by sandwich theorem), but*
$$\lim_{x\to 0} \cos\frac{1}{x}$$
   *does not exist. Hence the original limit, namely* $\lim\limits_{x\to 0} \frac{x^2 \sin\frac{1}{x}}{\sin x}$*, also fails to exist.*

   Is her solution correct or not? If not, what was wrong about it, and what is the correct solution then?