Solution 2

Section 6.2

2. Let f be defined on $[a,b]$ and $c \in [a,b]$. Then

   - c is a **critical point** of f if f' exists at c and $f'(c) = 0$.
   - c is a **relative maximum** (or **relative minimum**) of f if $\exists\ \delta > 0$ s.t. $f(c) \geq f(x)$ (or $f(c) \leq f(x)$) $\forall\ x \in [a,b] \cap (c-\delta, c+\delta)$.
   - A **relative extremum** is either a relative maximum or a relative minimum. Any differentiable relative extremum must be a critical point.

   To find relative extremum, there are 2 steps:

   (1) First, list all *critical points*, *non-differentiable points*, and *endpoints*. These are candidates for relative extrema.

   (2) Second, apply the first derivative test (Theorem 6.2.8) to the points in (1).

   It is always helpful to plot a graph first.

   (a) For $x \neq 0$, $f'(x) = 1 - \dfrac{1}{x^2} = 0 \quad \Rightarrow \quad x = \pm 1$.
   Hence $x = -1, 1$ are the critical points of f.
   Since any relative extremum must be a critical points when f is differentiable in its domain, apply the 1st derivative test to -1 and 1, i.e. for $x \neq 0$,

   $$f'(x) = 1 - \frac{1}{x^2} \begin{cases} > 0, & \text{for } x > 1 \text{ or } x < -1 \\ < 0, & \text{for } -1 < x < 1. \end{cases}$$

   Hence, relative maximum $= -1$, relative minimum $= 1$.
   The interval s.t. f is increasing $= (-\infty, -1] \cup [1, +\infty)$.
   The interval s.t. f is decreasing $= [-1, 0) \cup (0, 1]$.

   (b) relative maximum $= 1$, relative minimum $= -1$.
   The interval s.t. f is increasing $= [-1, 1]$.
   The interval s.t. f is decreasing $= (-\infty, -1] \cup [1, +\infty)$.

   (c) relative maximum $= 2/3$, no relative minimum.
   The interval s.t. f is increasing $= (0, 2/3]$.
   The interval s.t. f is decreasing $= [2/3, +\infty)$.

   (d) no relative maximum, relative minimum $= 1$.
   The interval s.t. f is increasing $= [1, +\infty)$.
   The interval s.t. f is decreasing $= (-\infty, 0) \cup (0, 1]$.

3. (a) For $x \neq \pm 1$, $f'(x) = 2x\ \mathrm{sgn}(x^2-1) = 0 \quad \Rightarrow \quad x = 0$
   Hence $x = 0$ is the critical point of f.
   For $x \neq \pm 1$, $\dfrac{f(x) - f(\pm 1)}{x - (\pm 1)} = \dfrac{|x^2-1|}{x-(\pm 1)} = |x \pm 1|\mathrm{sgn}[x - (\pm 1)]$
   $\Rightarrow \quad f'_+(\pm 1) = \lim\limits_{x \to \pm 1^+} |x \pm 1|\mathrm{sgn}[x-(\pm 1)] = 2,$
   $\quad\quad f'_-(\pm 1) = \lim\limits_{x \to \pm 1^-} |x \pm 1|\mathrm{sgn}[x-(\pm 1)] = -2$
   Hence $x = -1, 1$ are the non-differentiable points of f.
   And $x = -4, 4$ are the endpoints. All possible relative extrema are $0, \pm 1, \pm 4$. Apply the 1st derivative test to $0, \pm 1, \pm 4$, i.e. for $x \neq \pm 1$,

$$\mathrm{sgn} f'(x) = \mathrm{sgn}(x+1)\ \mathrm{sgn}\ x\ \mathrm{sgn}(x-1) \begin{cases} > 0, & \text{for } x > 1 \text{ or } -1 < x < 0 \\ < 0, & \text{for } x < -1 \text{ or } 0 < x < 1. \end{cases}$$

   Hence, relative maximum $= 0, \pm 4$, relative minimum $= \pm 1$.

   (b) no critical point, non-differentiable point $= 1$, endpoints $= 0,\ 2$.
   relative maximum $= 1$, relative minimum $= 0,\ 2$.

   (c) For $x \neq \pm\sqrt{12}$, $h'(x) = |x^2 - 12| + x\ \mathrm{sgn}(x^2-12)(2x) = 3\ \mathrm{sgn}(x^2-12)(x^2-4)$.
   critical point $= 2$, non-differentiable points $= \pm\sqrt{12} \notin [-2, 3]$, endpoints $= -2, 3$.
   relative maximum $= 2$, relative minimum $= -2, 3$.

   (d) For $x \neq 8$, $k'(x) = (x-8)^{1/3} + \dfrac{x}{3}(x-8)^{-2/3} = \dfrac{4}{3}(x-8)^{-2/3}(x-6)$.
   critical point $= 6$, non-differentiable point $= 8$, endpoints $= 0,\ 9$.
   relative maximum $= 0,\ 9$, relative minimum $= 6$,
   $x = 8$ is neither a relative maximum nor a relative minimum.

5. Let $f(x) := x^{1/n} - (x-1)^{1/n}$, for $x \geq 1$.
   Then $f'(x) = \dfrac{1}{n}x^{1/n-1} - \dfrac{1}{n}(x-1)^{1/n-1}$ for $x > 1$.
   Define $g(t) := t^{1/n-1}$ for $t > 0$, $g'(t) = \left(\dfrac{1}{n} - 1\right)\ t^{1/n-2} < 0$ since $n \geq 2$.
   Then for $x > 1$, $f'(x) = \dfrac{1}{n}g(x) - \dfrac{1}{n}g(x-1) < 0$. Hence f is strictly decreasing for $x > 1$.
   Note $a > b > 0$, then $a/b > 1$, hence $f(a/b) < \lim\limits_{x \to 1^+} f(x) = f(1)$, by continuity,
   i.e. $\left(\dfrac{a}{b}\right)^{1/n} - \left(\dfrac{a}{b} - 1\right)^{1/n} < 1 - (1-1) = 1 \quad \Rightarrow \quad a^{1/n} - b^{1/n} < (a-b)^{1/n}$.

6. Note that the function $f(t) := \sin t$ is differentiable on $\mathbb{R}$, with $f'(t) = \cos t$. In particular, given any $x, y \in \mathbb{R}$ with $x < y$, the function $f(t)$ is continuous on $[x, y]$, and differentiable on (x, y). Hence the mean-value theorem applies, from which we conclude that there exists some $c \in (x, y)$ such that

$$\sin x - \sin y = (\cos c)\,(x - y).$$

   Now just put absolute values on both sides, and observe that $|\cos c| \leq 1$. Then

$$|\sin x - \sin y| \leq |x - y|,$$

   as desired.

7. Note that the function $f(t) := \ln t$ is differentiable on $(0, \infty)$, with $f'(t) = 1/t$. In particular, given any $x \in (0, \infty)$ with $x > 1$, the function $f(t)$ is continuous on $[1, x]$, and differentiable on $(1, x)$. Hence the mean-value theorem applies, from which we conclude that there exists some $c \in (1, x)$ such that

$$\ln x - \ln 1 = \frac{x-1}{c}.$$

Now just observe that

$$\frac{1}{x} < \frac{1}{c} < 1,$$

since $c \in (1, x)$. Since $x - 1 > 0$, it follows that

$$\frac{x-1}{x} < \ln x - \ln 1 < 1 \cdot (x-1),$$

i.e.

$$\frac{x-1}{x} < \ln x < (x-1),$$

as desired.

9. For $x \neq 0$, $f(x) = 2x^4 + x^4 \sin \dfrac{1}{x} \geq 2x^4 - x^4 = x^4 > 0 = f(0)$
Hence f has an absolute minimum at $x = 0$.
For $x \neq 0$, $f'(x) = 8x^3 + 4x^3 \sin \dfrac{1}{x} + x^4 \cos \dfrac{1}{x} \left(-\dfrac{1}{x^2}\right) = x^2 \left(8x + 4x \sin \dfrac{1}{x} - \cos \dfrac{1}{x}\right)$
Define $a_n := 1/2n\pi$ and $b_n := 1/(2n\pi + \pi/2)$ with $\lim a_n = \lim b_n = 0$.
Then $f'(a_n) = \left(\dfrac{1}{2n\pi}\right)^2 \left(\dfrac{8}{2n\pi} - 1\right) < \left(\dfrac{1}{2n\pi}\right)^2 \left(\dfrac{8}{6n} - 1\right) < 0$ if $n \geq 2$
$f'(b_n) = \left(\dfrac{1}{2n\pi + \pi/2}\right)^2 \left(\dfrac{8}{2n\pi + \pi/2} - \dfrac{4}{2n\pi + \pi/2}\right) > 0 \;\; \forall\, n.$
Let $\varepsilon > 0$. Then $\exists\, N_1, N_2 \in \mathbb{N}$ s.t. $|a_{N_1}| < \varepsilon$ and $|b_{N_2}| < \varepsilon$, i.e. $a_{N_1}, b_{N_2} \in (-\varepsilon, \varepsilon)$.
WLOG assume $N_1 \geq 2$ *. Hence $f'(a_{N_1}) < 0, f'(b_{N_2}) > 0$ with $a_{N_1}, b_{N_2} \in (-\varepsilon, \varepsilon)\ \forall\, \varepsilon > 0$.
Hence the derivative has both positive and negative values in every nbd of 0.

* **Remark** We can replace N_1 by $\max(N_1, 2)$. Or we can interpret N_1 already chosen to be ≥ 2. This is a useful skill in analysis.

10. $\dfrac{g(x) - g(0)}{x - 0} = \dfrac{x + 2x^2 \sin(1/x)}{x} = 1 + 2x \sin \dfrac{1}{x} \quad \Rightarrow \quad g'(0) = 1 + 2 \lim\limits_{x \to 0} x \sin \dfrac{1}{x} = 1 + 2(0) = 1.$
For $x \neq 0, g'(x) = 1 + 4x \sin(\frac{1}{x}) - 2\cos(\frac{1}{x})$. Define $a_n := 1/2n\pi$ and $b_n := 1/(2n\pi + \pi/2)$ with $\lim a_n = \lim b_n = 0$.
Then $g'(a_n) = 1 - 2\cos 2n\pi = -1 < 0$, and
$g'(b_n) = 1 + 4(\dfrac{1}{2n\pi + \frac{\pi}{2}}) > 0.$

Let $\varepsilon > 0$. Then $\exists\, N_1, N_2 \in \mathbb{N}$ s.t. $|a_{N_1}| < \varepsilon$ and $|a_{N_2}| < \varepsilon$, i.e. $a_{N_1}, b_{N_2} \in (-\varepsilon, \varepsilon)$.
Hence $g'(a_{N_1}) > 0, g'(b_{N_2}) < 0$ with $a_{N_1}, b_{N_2} \in (-\varepsilon, \varepsilon)\ \forall\, \varepsilon > 0$.
Thus g cannot be monotonic on $(-\varepsilon, \varepsilon)\ \forall\, \varepsilon > 0$, (read Theorem 6.2.7 carefully), i.e. any nbd of 0.

11. Take $f(x) := \sqrt{x}$ is continuous on $[0,1]$ and hence uniformly continuous on $[0,1]$.
For $x > 0$, $f'(x) = \dfrac{1}{2\sqrt{x}}$ is unbounded, which can be proved by putting $x = x_n := \dfrac{1}{4n^2} \to 0$.

13. Let $x, y \in I$ s.t. $x < y$. By MVT, $\exists\ \xi \in (x,y)$ s.t.
$f(x) - f(y) = f'(\xi)(x-y) < 0$, as in particular, $f'(\xi) > 0$ and $x - y < 0$
$\Rightarrow \quad f(x) < f(y)$.
Hence f is strictly increasing on I.

18. Let $\varepsilon > 0$. Then $\exists\ \delta$ s.t.
$$\left|\frac{f(x)-f(c)}{x-c} - f'(c)\right| < \varepsilon, \quad \forall\ 0 < |x-c| < \delta.$$
For $x < c < y$ inside $(c-\delta, c+\delta)$,
$$-\varepsilon(y-c) < f(y) - f(c) - f'(c)(y-c) < \varepsilon(y-c)$$
$$-\varepsilon(x-c) > f(x) - f(c) - f'(c)(x-c) > \varepsilon(x-c)$$
$$-\varepsilon(y-x) < f(y) - f(x) - f'(c)(y-x) < \varepsilon(y-x)$$
$$\left|\frac{f(y)-f(x)}{y-x} - f'(c)\right| < \varepsilon.$$

Supplementary Exercise

1. Separating the whole interval into a sequence of finite intervals: $(1,2), \cdots, (2^k, 2^{k+1}), \cdots$.
All of them satisfy the assumption of Mean Value Theorem. Hence we get:
$f'(x_1) = \frac{f(2)-f(1)}{2^0}$
$f'(x_2) = \frac{f(4)-f(2)}{2^1}$
$\vdots$
$f'(x_{k+1}) = \frac{f(2^{k+1})-f(2^k)}{2^k}$
$\vdots$

$|f'(x_{k+1}| \le 2M/2^k = 2^{-k+1}M \to 0$ as $k \to \infty$
Since $x_{k+1} \ge 2^k$, so $x_k \to \infty$ as $k \to \infty$
So we have the sequence.