MATH 2060 Mathematical Analysis II
Tutorial Class 5
Lee Man Chun

1. (a) Show that if $f \in R[a,b]$, then for any sequence of tagged partition $\dot{P}_n$ of $[a,b]$, $||P_n|| \to 0$ implies $S(f, \dot{P}_n) \to \int_a^b f$ as $n \to \infty$.

   (b) Find the following limits.

   i. $\lim_{n\to\infty} \sum_{k=0}^{n} \frac{1}{k+n}$.

   ii. $\lim_{n\to\infty} \left[\frac{n^2}{n^2+1} \cdot \frac{n^2}{n^2+2^2} \cdot \frac{n^2}{n^2+3^2} \cdots \frac{n^2}{n^2+n^2}\right]^{\frac{1}{n}}$.

2. Let $f : [a,b] \to \mathbb{R}$ be Riemann integrable and $g : \mathbb{R} \to \mathbb{R}$ be continuous. Show that $g \circ f$ is Riemann integrable on $[a,b]$.

3. Let $f : [a,b] \to \mathbb{R}$ be a bounded function at which $f \in R[c,b]$ for any $c > a$. Prove that $f \in R[a,b]$ and $\int_a^b f = \lim_{c\to a^+} \int_c^b f$.

4. (a) Let $g \in R[a,b]$ and $f : [a,b] \to \mathbb{R}$ be a continuous function. Suppose $g \geq 0$ on $[a,b]$. Show that there exists $c \in [a,b]$ such that $\int_a^b fg = f(c) \int_a^b g$.

   (b) Let $f : [0, +\infty) \to \mathbb{R}$ be a continuous function with $\lim_{x\to\infty} f(x) = L \in \mathbb{R}$. Suppose $\{a_n\}, \{b_n\}$ are two sequence in $\mathbb{R}^+$ such that $a_n \to 0, b_n \to \infty$ as $n \to \infty$. Show that for all $0 < r < s$,

$$\lim_{n\to\infty} \int_{a_n}^{b_n} \frac{f(rx) - f(sx)}{x} = (f(0) - L) \log \frac{s}{r}.$$

5. Let $f : [a,b] \to \mathbb{R}$ be a continuous function at which $f(x) \geq 0$. Show that

$$\lim_{n\to\infty} (\int_a^b f^n)^{\frac{1}{n}} = \sup\{f(x) : x \in [a,b]\}.$$