Suggested solution of HW3

P.215 Q2:

Let $P : 0 = x_0 < x_1 < ... < x_n = 1$ be a partition on $[0,1]$. On each interval $[x_i, x_{i+1}]$, since $\mathbb{Q}$ is dense in $\mathbb{R}$, $M_i = x_{i+1} + 1 > 1$ and $m_i = 0$.
So, we have

$$U(f,P) = \sum_{i=1}^{n} M_i \Delta x_i > \sum_{i=1}^{n} \Delta x_i = 1.$$

$$L(f,P) = \sum_{i=1}^{n} m_i \Delta x_i = 0.$$

Hence, h is not Riemann integrable as

$$\sup_P L(f,P) = 0 \ < \ 1 \ \leq \inf_P U(f,P).$$

P.215 Q8:

Suppose not, there exists $x_0 \in [a,b]$ such that $f(x_0) > 0$. By continuity of f, there exists $\delta > 0$ such that

$$f(x) > \frac{f(x_0)}{2} \ , \forall \ |x - x_0| < \delta \text{ and } x \in [a,b].$$

Let $I = [a,b] \cap V_\delta(x_0)$, without loss of generality, we can assume δ small enough such that $|I| \geq \delta$.

$$0 = \int_a^b f \geq \int_I f > \frac{f(x_0)}{2} |I| > 0.$$

Contradiction arise.

P.215 Q18:

By Max-Min theorem, there exists $c \in [a,b]$ such that $f(c) = M = \sup\{f(x) : x \in [a,b]\}$. Let $\epsilon > 0$ be given, there exists $\delta > 0$ such that

$$M \geq f(x) > M - \epsilon \ , \forall \ |x - c| < \delta \text{ and } x \in [a,b].$$

Let $I = [a,b] \cap V_\delta(c)$, without loss of generality, we can assume δ small enough such that $|I| \geq \delta$. For any $n \in \mathbb{N}$,

$$(M - \epsilon) \ \delta^{\frac{1}{n}} < \left(\int_I f^n \right)^{\frac{1}{n}} \leq \left(\int_a^b f^n \right)^{\frac{1}{n}} \leq M(b-a)^{\frac{1}{n}}.$$

Since, $\delta^{\frac{1}{n}}$ and $(b-a)^{\frac{1}{n}}$ goes to 1, as n goes to infinity. We have

$$M - \epsilon \leq \liminf_n \left(\int_a^b f^n \right)^{\frac{1}{n}} \leq \limsup_n \left(\int_a^b f^n \right)^{\frac{1}{n}} \leq M.$$

The above inequality hold for any fixed $\epsilon > 0$ which imply

$$\lim_n \left(\int_a^b f^n \right)^{\frac{1}{n}} = M.$$