## Introduction

The link between inequalities and intervals is made, among other things, by absolute value:

$$\mathrm{abs}(x) = \begin{cases} x, & x \geq 0; \\ \\ -x, & x < 0. \end{cases}$$

Traditionally, we denote $\mathrm{abs}(x)$ by $|x|$.

The absolute value (or 'absolute value function') is also known as 'modulus', 'norm' or 'length'. It satisfies the following 'triangle inequlaity'. ($\Delta$-inequality.)

**Theorem**
$|a+b| \leq |a| + |b|$ (or more precisely, 'for all real nos. $a$, $b$, the above-mentioned inequality holds').

**Proof.** We need the identity $|a \cdot b| = |a||b|$.

We then prove this theorem by first squaring both $|a+b|$ to get

$$|a+b|^2 = (a+b)^2$$

$= a^2 + 2ab + b^2 = |a|^2 + 2ab + |b|^2$ (because $x^2 = |x|^2$)
$\leq |a|^2 + 2|ab| + |b|^2$ (by $x \leq |x|$)
$= |a|^2 + 2|a||b| + |b|^2$ (by $|ab| = |a||b|$)
$= (|a| + |b|)^2$
implying $|a+b| \leq |a| + |b|$. $\square$

**Corollary.** $||a| - |b|| \leq |a-b|$.
**Proof.** $|a| = |a-b+b| \leq |a-b| + |b|$
Hence $|a| \leq |a-b| + |b|$
Letting $c = -|b|$ in the above inequality and adding $c$ to both sides of the inequality sign, we obtain
$|a| + (-|b|) \leq |a-b| + |b| - |b|$
which gives $|a| - |b| \leq |a-b|$.
Next, interchanging the roles of $a$ and $b$ gives
$|b| - |a| \leq |b-a|$.
Combining these two inequalities, we have proved
$||a| - |b|| \leq |a-b|$ $\square$

## Connection between $|\cdot|$ and Interval

The reason why we learned inequalities is because we'll use them to describe limits of sequences (and later 'functions') rigorously.

Before we talk about limits, let's see how mathematicians describe 'closeness'.

Suppose we are given a real no. $L$, how do we describe numbers which are arbitrarily (i.e. as close as we wish) close (but not equal) to $L$?

Just for the purpose of illustration, suppose $L = 2$, and we want to describe all numbers close to 2 with an error at most 0.01, then all we need is the open interval

$$(2 - 0.01, 2 + 0.01).$$

If now the error is 0.001, then the interval becomes

$$(2 - 0.001, 2 + 0.001).$$

The numbers 0.01 and 0.001 measure how close to 2 we want the numbers to be.

We denote this tolerance /error and denote it by $\varepsilon$.

Summarizing the above, if we want the error to be $\varepsilon$, then the interval is given by

$$(2 - \varepsilon, 2 + \varepsilon)$$

which can also be written (in inequality form) by:

$$\{x \in \mathbb{R}|\ |x - 2| < \varepsilon\}.$$

## Limit of Sequence

Recall that a sequence is a 'listed' (or 'indexed') list of numbers of the form

$$a_0, a_1, \cdots, a_n, \cdots$$

The numbering/indexing starts usually from 0 or 1, but it can also start from any finite no. $K$.

### Examples

(1) 1,1/2,1/3,$\cdots$, $1/n$, $\cdots$

(2) $1, -1, 1, -1, \cdots,$

One difference between the sequence in (1) and that in (2) is that the sequence in (1) converges to 0 (i.e. gets 'closer and closer' to zero), whereas the sequence in (2) converges to no limit.

In symbols, we denote it by either

(1) For some $L$, $n \to \infty$ implies $a_n \to L$; (the mathematical notation for the phrase 'for some' is: $\exists$)

(2) For each $L$, $n \to \infty$ does not imply $a_n \to L$ (the mathematical notation for the phrase 'for any' is: $\forall$)

**Remark.** Item (2) is not too easy to write down rigorously.

Instead of the above ways to describe approaching $L$ or not approaching $L$, many people also write the same thing down in the following form

(1) For some $L$, $\lim_{n\to\infty} a_n = L$;

(2) For each $L$, $\lim_{n\to\infty} a_n \neq L$.

**Remark.** The phrase 'for each' is also written 'for all' or 'for every' or 'for any' by some authors.
**Remark.** The phrase 'for some' is also written 'there exists' by some authors.

But what is a good definition of sequence such as (1) above (i.e. sequence with limit $L$)?

**Definition** ($\varepsilon - N$ defintion) Let $(a_n)$ (or $\{a_n\}$) be a sequence of real nos., then we say $\lim_{n\to\infty} a_n = L$ (for some $L$) if

For each given $\varepsilon > 0$, there exists (i.e. we need to find it whenever we can) a natural number $N$ such that for each natural number bigger than $N$, the following inequality is satisfied:

$$|a_n - L| < \varepsilon.$$

**Remark.** In a definition, the adjective 'if' usually means 'if and only if'.