Tutorial 5. Math2010E

Limit and continuity of multivariable functions.

$F: \mathbb{R}^n \to \mathbb{R}^m$

$F(x_1, \dots x_n) = (f_1(x_1 \dots x_n), \dots f_m(x_1, \dots x_n))$

$\forall i.\ f_i(x_1 \dots x_n)$ is a multivariable functions

$\lim_{\vec{x} \to \vec{x_0}} F = (\lim f_1, \dots, \lim f_m)$

therefore, to study $\lim f_i$ is the first step.

Recall: In one-variable case

$g = g(x): \mathbb{R} \to \mathbb{R}$.

e.g. $\lim_{x \to 0} \left(\frac{\cos x}{\cos 2x}\right)^{\frac{1}{x^2}}$.

> singular type: $(1)^{\infty}$

$\boxed{\text{model. } \lim_{x\to 0} (1+x)^{\frac{1}{x}} \left(\text{or } \lim_{x \to +\infty} (1+\frac{1}{x})^{x}\right)}$

~~Aim.~~ Let $\frac{\cos x}{\cos 2x} = 1 + y$. ($y$ is a function of x)

$$y = \frac{\cos x}{\cos 2x} - 1 = \frac{\cos x - \cos 2x}{\cos 2x}$$

Then $\left(\frac{\cos x}{\cos 2x}\right)^{\frac{1}{x^2}} = \cancel{\ln y + 1}\ (1+y)^{\frac{1}{y} \cdot y \cdot \frac{1}{x^2}}$

$\lim_{x \to 0} y = 0$.

$$\frac{y}{x^2} = \frac{\cos x - \cos 2x}{x^2 \cos 2x} = \frac{\cos x + 1 - 2\cos^2 x}{x^2 \cos 2x} = \frac{(2\cos x + 1)(1 - \cos x)}{\cos 2x \cdot x^2} = \frac{(2\cos x + 1)}{\cos 2x}\left(\frac{2\sin^2\frac{x}{2}}{x^2}\right)$$

$\cos 2x = \cos^2 x - \sin^2 x = 2\cos^2 x - 1$

$\begin{matrix} & 2 & & 1 \\ -1 & & \times & \\ & & & 1 \end{matrix}$

$\cos x = \cos^2\frac{x}{2} - \sin^2\frac{x}{2} = 1 - 2\sin^2\frac{x}{2}$.

therefore

$$\lim_{x\to 0} \frac{y}{x^2} = \lim_{x \to 0} \frac{2+1}{1} \cdot \frac{2 \cdot (\frac{x}{2})^2}{(x)^2} = \frac{3}{2}.$$

$\left(\lim_{x \to 0} \frac{\sin\frac{x}{2}}{x} = 2\lim_{x\to 0} \frac{\sin\frac{x}{2}}{\frac{x}{2}} = 2\right)$

Hence

$$\lim_{x \to 0} \left(\frac{\cos x}{\cos 2x}\right)^{\frac{1}{x^2}} = \lim_{y \to 0} (1+y)^{\frac{1}{y} \cdot \lim_{x\to 0} y \cdot \frac{1}{x^2}} = e^{\frac{3}{2}}$$

Remark:

Strictly Speaking, it should be.

$$\lim_{x\to0}\left(\frac{\cos x}{\cos 2x}\right)^{\frac{1}{x^2}} = \lim_{x\to0}\exp\left(y\cdot\frac{1}{x^2}\log(1+y)^{\frac{1}{y}}\right)$$

$$= \cdots \exp\left(\lim_{x\to0}\left(y\cdot\frac{1}{x^2}\right)\cdot\lim_{x\to0}\log(1+y)^{\frac{1}{y}}\right)$$

continuity of exponential function.

$$= \exp\left[\lim_{x\to0}\left(y\cdot\frac{1}{x^2}\right)\cdot\log\lim_{x\to0}(1+y)^{\frac{1}{y}}\right]$$

continuity of logarithm function.

$$= \left[\lim_{y\to0}(1+y)^{\frac{1}{y}}\right]^{\lim_{x\to0}\left(y\cdot\frac{1}{x^2}\right)},$$

#.

Recall. All singular types and models.

1> $(1)^{\infty}$ $\qquad \lim_{x\to0}(1+x)^{\frac{1}{x}}$.

3> $(\infty)^0 = \exp\left[\log(\infty^0)\right]$ ~~than~~
$= \exp(0\cdot\cancel{\log}\,\infty)$

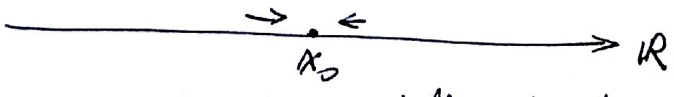
2> $\infty\cdot0$ $\qquad \begin{cases}\lim_{x\to0}\frac{1}{x}\cdot\sin x \\ \lim_{x\to0}\frac{x^m}{x^n}.\end{cases}$

4> $0^0 = \exp(0\log0) = \exp(0\cdot\infty)$

$$\lim_{x\to x_0}f(x) = \lim_{x\to x_0^+}f(x) = \lim_{x\to x_0^-}f(x)$$

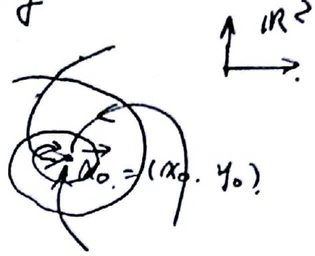
Characteristic.
① left limit Exists and finite
② Right limit Exists and finite
③ Left limit = Right limit

~~Under~~ Behind this Characteristic. is due to the fact IR is a 1-D space



one can only approach x_0 via two different directions left & Right.

However, in any higher dimensional space (2-D, for example) to approach a point there are (in-fact) infinite many path.



Now, to clear "approach", it means the distance between the given point and the approximating ~~poi~~ sequences tend to zero.

i.e (in 2D) $\lim\limits_{\substack{x \to x_0 \\ y \to y_0}} f(x, y) := \lim\limits_{\underbrace{\sqrt{(x-x_0)^2 + (y-y_0)^2}}_{\text{dist}((x,y),\,(x_0,y_0))} \to 0} f(x, y)$

by this definition, the right-hand side is acturally a limit w.r.t one variable. (distance)

One ~~Strateg~~ Strategy is to bdd the function via a ~~dis~~ function of the distance. function.

Example. $\lim\limits_{\substack{x \to 0 \\ y \to 0}} (x^2+y^2)^{x^2y^2}$

$\boxed{\text{We only consider when } 0 < x^2+y^2 < 1}$

Note: $0 \le x^2y^2 \le \dfrac{x^4+y^4}{2} \le \dfrac{(x^2+y^2)^2}{2} \quad \left[(x^2-y^2)^2 \ge 0 \right]$

$(x^2+y^2)^0 \ge (x^2+y^2)^{x^2y^2} = \exp\left[x^2y^2 \cdot \log(x^2+y^2) \right]$

$\ge \exp\left[\dfrac{x^4+y^4}{2} \log(x^2+y^2) \right]$

$\ge \exp\left[\dfrac{(x^2+y^2)^4}{2} \log(x^2+y^2) \right]$

$= \exp\left[\dfrac{d^8}{2} \log(d^2) \right]$

$1 \ge \lim\limits_{\substack{x \to 0 \\ y \to 0}} (x^2+y^2)^{x^2y^2} \ge \exp \lim\limits_{d \to 0} \left[\dfrac{d^8}{2} \log d^2 \right] = \exp[0] = 1$

"Sanwich Theorem" $\Rightarrow \lim\limits_{\substack{x \to 0 \\ y \to 0}} (x^2+y^2)^{x^2y^2} = 1$

Example. $\lim_{\substack{x\to 0\\ y\to 0}} (x^2+y^2)^{x\cdot y}$

$$-\left(\frac{x^2+y^2}{2}\right) \le |x\cdot y| \le \frac{x^2+y^2}{2}$$

$$\lim_{\substack{x\to 0\\ y\to 0}} \exp\left[\frac{x^2+y^2}{2}\log(x^2+y^2)\right]$$

$$\le \lim_{\substack{x\to 0\\ y\to 0}} (x^2+y^2)^{x\cdot y} = \lim_{\substack{x\to 0\\ y\to 0}} \exp\left(x\cdot y\log(x^2+y^2)\right)$$

$$\le \lim_{\substack{x\to 0\\ y\to 0}} \exp\left[-\left(\frac{x^2+y^2}{2}\right)\log(x^2+y^2)\right]$$

Similarly $\lim_{\substack{x\to 0\\ y\to 0}} (x^2+y^2)^{xy} = 1$

Remark: Polar Representation.

Limit Does Not exist?

Basic Strategy: find two (different) sequences with different limits.

Example
$\lim_{\substack{x\to 0\\ y\to 0}} f(x,y)$, $\lim_{x\to 0}\left(\lim_{y\to 0} f(x,y)\right)$, $\lim_{y\to 0}\left(\lim_{x\to 0} f(x,y)\right)$.

a>. $f(x,y) = \dfrac{x^2y^2}{x^2y^2 + (x-y)^2}$.

Sequence 1. $x=y$. $\lim_{x\to 0} f(x,x) = 1$.

Sequence 2 $x=-y$ $\lim_{x\to 0} f(x,-x) = 0$

[~~Inform~~ or. $x = ky$] $\lim_{\substack{x\to 0\\ y\to 0}} f(x,y)$ Not exist.

However. for $x\neq 0$. $\lim_{y\to 0} f(x,y) = 0$. $\Rightarrow \lim_{x\to 0}\left(\lim_{y\to 0} f(x,y)\right) = 0$

Similarly $\lim_{y\to 0}\left(\lim_{x\to 0} f(x,y)\right) = 0$.

b>. $f(x,y) = (x+y)\sin\frac{1}{x}\sin\frac{1}{y}$.

$$|f(x,y)| \le |x+y| \le |x|+|y| \le 2\sqrt{x^2+y^2}$$

$$\lim_{\substack{x\to 0\\ y\to 0}} |f(x,y)| \le \lim_{\substack{x\to 0\\ y\to 0}} 2\sqrt{x^2+y^2} = 0$$

$$\Rightarrow \lim_{\substack{x\to 0\\ y\to 0}} f(x,y) = 0. \quad (\text{Sanwich Theorem}).$$

But $\lim_{x\to 0}\left(\lim_{y\to 0} f(x,y)\right)$ and $\lim_{y\to 0}\left(\lim_{x\to 0} f(x,y)\right)$ Not Exist.