MATH 2010E ADVANCED CALCULUS I
MID-TERM EXAMINATION SOLUTIONS

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Please work on FIVE of the following SIX problems. If you work on all SIX, please cross out the problem you do NOT want to be graded, or else, only the first FIVE problems you worked on will be graded.

Problem 1. (20 points)
(a) Find an equation of the plane passing through the points $(-6,8,9)$ and $(-9,19,0)$ and orthogonal to the plane $2x-7y+5z=12$.
(b) Find a parametric form of the straight line given by the two equations

$$\begin{cases} 2x+y-3z=3 \\ x+2y+3z=6. \end{cases}$$

Solution.
(a) Let $\mathbf{n}$ be a normal vector of the desired plane. Then $\mathbf{n}$ is orthogonal to both

$$(-6,8,9)-(-9,19,0)=(3,-11,9) \qquad \text{and} \qquad (2,-7,5).$$

Hence,

$$\mathbf{n}=\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & -11 & 9 \\ 2 & -7 & 5 \end{vmatrix} = \begin{vmatrix} -11 & 9 \\ -7 & 5 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 3 & 9 \\ 2 & 5 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 3 & -11 \\ 2 & -7 \end{vmatrix}\mathbf{k} = (8,3,1)$$

Therefore, the desired plane is $8x+3y+z=D$ such that $(-6,8,9)$ lies on this plane, i.e.

$$D=8(-6)+3(8)+9=-15.$$

$\therefore$ The desired plane is $8x+3y+z=-15$.

(b) First, we perform Gauss-Jordan elimination.

$$\left(\begin{array}{ccc|c} 2 & 1 & -3 & 3 \\ 1 & 2 & 3 & 6 \end{array}\right) \xrightarrow{R_1 \leftrightarrow R_2} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 2 & 1 & -3 & 3 \end{array}\right) \xrightarrow{R_2-2R_1} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 0 & -3 & -9 & -9 \end{array}\right)$$

$$\xrightarrow{\frac{1}{-3}R_2} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 0 & 1 & 3 & 3 \end{array}\right) \xrightarrow{R_1-2R_2} \left(\begin{array}{ccc|c} 1 & 0 & -3 & 0 \\ 0 & 1 & 3 & 3 \end{array}\right).$$

Hence, the solution set is $\{(3t, 3-3t, t) : t \in \mathbb{R}\}$. Therefore, a parametric form of the straight line is

$$\{(0,3,0)+t(3,-3,1) : t \in \mathbb{R}\}.$$

□

Problem 2. (20 points)
(a) Let $\mathbf{u}$ and $\mathbf{v}$ be two vectors in $\mathbb{R}^3$. Show that $\|\mathbf{u}\|\mathbf{v}+\|\mathbf{v}\|\mathbf{u}$ is an angle bisector between $\mathbf{u}$ and $\mathbf{v}$.
(b) Let P_1 be the plane $-x+2y+2z=4$ and P_2 be the plane $3x+4y-12z=-28$. Find an equation of a plane that bisects an angle between P_1 and P_2.

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(*Hint*: The answer may not be unique. You only need to give ONE equation of ONE such plane.)

Solution.
(a) Let $\mathbf{x} = \|\mathbf{u}\|\mathbf{v} + \|\mathbf{v}\|\mathbf{u}$. Let α be an angle between $\mathbf{u}$ and $\mathbf{x}$, and β an angle between $\mathbf{v}$ and $\mathbf{x}$, where $\alpha, \beta \in [-\pi, \pi]$.

$$\|\mathbf{x}\| \cos\alpha = \frac{\mathbf{u}}{\|\mathbf{u}\|} \cdot \mathbf{x} = \mathbf{u}\cdot\mathbf{v} + \|\mathbf{v}\|\|\mathbf{u}\|, \quad \text{and} \quad \|\mathbf{x}\| \cos\beta = \frac{\mathbf{v}}{\|\mathbf{v}\|} \cdot \mathbf{x} = \|\mathbf{u}\|\|\mathbf{v}\| + \mathbf{v}\cdot\mathbf{u}.$$

Hence, $\cos\alpha = \cos\beta$, or $\alpha = \pm\beta$. Geometrically, the angle between $\mathbf{u}$ and $\mathbf{x}$ is the same as the angle between $\mathbf{v}$ and $\mathbf{x}$, i.e. $\mathbf{x}$ is the angle bisector between $\mathbf{u}$ and $\mathbf{v}$.

(b) Let $\mathbf{n}$ be a normal vector of a desired plane. Then $\mathbf{n}$ is an angle bisector between the normal vectors of the planes P_1 and P_2. From (a),

$$\mathbf{n} = \sqrt{(-1)^2 + 2^2 + 2^2} \cdot (3, 4, -12) + \sqrt{3^2 + 4^2 + (-12)^2} \cdot (-1, 2, 2) = (-4, 38, -10).$$

To find an intersection point between P_1 and P_2, we perform Gauss-Jordan elimination.

$$\left(\begin{array}{ccc|c} -1 & 2 & 2 & 4 \\ 3 & 4 & -12 & -28 \end{array}\right) \xrightarrow{-R_1} \left(\begin{array}{ccc|c} 1 & -2 & -2 & -4 \\ 3 & 4 & -12 & -28 \end{array}\right) \xrightarrow{R_2 - 3R_1} \left(\begin{array}{ccc|c} 1 & -2 & -2 & -4 \\ 0 & 10 & -6 & -16 \end{array}\right)$$
$$\xrightarrow{\frac{1}{10}R_2} \left(\begin{array}{ccc|c} 1 & -2 & -2 & -4 \\ 0 & 1 & -0.6 & -1.6 \end{array}\right) \xrightarrow{R_1 + 2R_2} \left(\begin{array}{ccc|c} 1 & 0 & -3.2 & -7.2 \\ 0 & 1 & -0.6 & -1.6 \end{array}\right).$$

Hence, an intersection point between P_1 and P_2 is $(-4, -1, 1)$. Therefore, a desired plane is $-4x + 38y - 10z = D$ such that $(-4, -1, 1)$ lies on this plane, i.e.

$$D = -4(-4) + 38(-1) - 10(1) = -32.$$

$\therefore$ A desired plane is $-4x + 38y - 10z = -32$ (or $2x - 19y + 5z = 16$).

(*The equation of another desired plane is* $11x - 7y - 31z = -68$, *whose normal vector can be obtained by the cross product between* $\mathbf{n}$ *and the vector of the intersection line.*)

□

Problem 3. (20 points)
Let $f(x, y, z) = 9x^2 - 3y^2 - 3z^2 - 8xy + 8xz + 4yz$. Determine the shape of the level surface given by $f(x, y, z) = -5$.

Solution.

$$\begin{aligned} \begin{vmatrix} 9-\lambda & -4 & 4 \\ -4 & -3-\lambda & 2 \\ 4 & 2 & -3-\lambda \end{vmatrix} &= (9-\lambda)(-3-\lambda)(-3-\lambda) + (-4)(2)(4) + (4)(-4)(2) \\ &\quad - (4)(-3-\lambda)(4) - (2)(2)(9-\lambda) - (-4)(-4)(-3-\lambda) \\ &= -\lambda^3 + 3\lambda^2 + 45\lambda + 81 - 64 + 48 + 16\lambda - 36 + 4\lambda + 48 + 16\lambda \\ &= -\lambda^3 - 3\lambda^2 - 81\lambda - 77 \\ &= (-\lambda - 1)(\lambda + 7)(\lambda - 11). \end{aligned}$$

Hence, the three eigenvalues are -7, -1, and 11. In other words, the level surface is

$$\frac{u^2}{a^2} - \frac{v^2}{b} - \frac{w^2}{c} = -5.$$

This is a nondegenerated hyperboloid of one sheet.

□

Problem 4. (20 points)
Let
$$\mathbf{x}(t) = \left(\ln t - \frac{1}{2}\ln(t^2+2),\ \frac{1}{2}\ln(t^2+2),\ \sqrt{2}\tan^{-1}\frac{t}{\sqrt{2}}\right)$$
be a curve in $\mathbb{R}^3$, where $t > 0$.
(a) Find the length of the curve for $t \in [1, a]$.
(b) Reparametrize $\mathbf{x}(t)$ into $\widetilde{\mathbf{x}}(s)$ such that $\|\widetilde{\mathbf{x}}'(s)\| = 1$ for all s.
(*Hint*: $\dfrac{d}{dx}\tan^{-1}x = \dfrac{1}{x^2+1}$.)

Solution.
(a)
$$\mathbf{x}'(t) = \left(\frac{1}{t} - \frac{2t}{2(t^2+2)},\ \frac{2t}{2(t^2+2)},\ \sqrt{2}\frac{1}{\frac{t^2}{2}+1}\frac{1}{\sqrt{2}}\right) = \left(\frac{1}{t} - \frac{t}{t^2+2},\ \frac{t}{t^2+2},\ \frac{2}{t^2+2}\right),$$

$$\begin{aligned}\|\mathbf{x}'(t)\| &= \sqrt{\left(\frac{1}{t} - \frac{t}{t^2+2}\right)^2 + \left(\frac{t}{t^2+2}\right)^2 + \left(\frac{2}{t^2+2}\right)^2} \\ &= \sqrt{\frac{1}{t^2} - \frac{2t}{t(t^2+2)} + \frac{t^2}{(t^2+2)^2} + \frac{t^2}{(t^2+2)^2} + \frac{4}{(t^2+2)^2}} \\ &= \sqrt{\frac{1}{t^2} + \frac{-2(t^2+2)+2t^2+4}{(t^2+2)^2}} = \frac{1}{t} \qquad \text{for } t > 0.\end{aligned}$$

Therefore, the length of the curve is
$$\int_1^a \|\mathbf{x}'(t)\|dt = \ln t|_1^a = \ln a.$$
(b) From (a), the arc length parameter is
$$s = \int_1^t \|\mathbf{x}'(\tau)\|d\tau = \ln t.$$
Hence, $t(s) = e^s$. Therefore,
$$\begin{aligned}\widetilde{\mathbf{x}}(s) = \mathbf{x}(t(s)) &= \left(\ln e^s - \frac{1}{2}\ln(e^{2s}+2),\ \frac{1}{2}\ln(e^{2s}+2),\ \tan^{-1}\frac{e^{2s}}{\sqrt{2}}\right) \\ &= \left(s - \frac{1}{2}\ln(e^{2s}+2),\ \frac{1}{2}\ln(e^{2s}+2),\ \tan^{-1}\frac{e^{2s}}{\sqrt{2}}\right).\end{aligned}$$

□

Problem 5. (20 points)
(a) Let $\mathbf{x}(t) = (e^t\cos t, e^t\sin t, e^t)$. Find $\mathbf{T}(t)$, $\mathbf{N}(t)$, $\mathbf{B}(t)$, and $\kappa(t)$.
(b) Let $\mathbf{x}(t) = (\sin t\cos t, \cos^2 t, \sin t)$. Without computing $\mathbf{x}'(t)$, justify that $\mathbf{x}'(t)$ is orthogonal to $\mathbf{x}(t)$.

Solution.
(a) $\mathbf{x}'(t) = (e^t\cos t - e^t\sin t, e^t\sin t + e^t\cos t, e^t) = e^t(\cos t - \sin t, \sin t + \cos t, 1)$.
$$\begin{aligned}\|\mathbf{x}'(t)\| &= e^t\sqrt{(\cos t - \sin t)^2 + (\sin t + \cos t)^2 + 1^2} \\ &= e^t\sqrt{\cos^2 t - 2\cos t\sin t + \sin^2 t + \sin^2 t + 2\sin t\cos t + \cos^2 t + 1} = e^t\sqrt{3}.\end{aligned}$$
$$\mathbf{T}(t) = \frac{e^t(\cos t - \sin t, \sin t + \cos t, 1)}{e^t\sqrt{3}} = \frac{(\cos t - \sin t, \sin t + \cos t, 1)}{\sqrt{3}}.$$

$\dfrac{d\mathbf{T}}{dt} = \dfrac{(-\sin t - \cos t, \cos t - \sin t, 0)}{\sqrt{3}}.$

$\left\| \dfrac{d\mathbf{T}}{dt} \right\| = \dfrac{1}{\sqrt{3}} \sqrt{(-\sin t - \cos t)^2 + (\cos t - \sin t)^2} = \sqrt{\dfrac{2}{3}}.$

$\mathbf{N}(t) = \dfrac{(-\sin t - \cos t, \cos t - \sin t, 0)/\sqrt{3}}{\sqrt{2/3}} = \dfrac{(-\sin t - \cos t, \cos t - \sin t, 0)}{\sqrt{2}}.$

$$\mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\cos t - \sin t}{\sqrt{3}} & \dfrac{\sin t + \cos t}{\sqrt{3}} & \dfrac{1}{\sqrt{3}} \\ \dfrac{-\sin t - \cos t}{\sqrt{2}} & \dfrac{\cos t - \sin t}{\sqrt{2}} & 0 \end{vmatrix}$$

$$= \left(\frac{\sin t - \cos t}{\sqrt{6}}, -\frac{\sin t + \cos t}{\sqrt{6}}, \frac{(\cos t - \sin t)^2 + (\sin t + \cos t)^2}{\sqrt{6}} \right)$$

$$= \frac{(\sin t - \cos t, -\sin t - \cos t, 2)}{\sqrt{6}}.$$

$\kappa(t) = \dfrac{1}{\|\mathbf{x}'(t)\|} \left\| \dfrac{d\mathbf{T}}{dt} \right\| = \dfrac{1}{e^t \sqrt{3}} \sqrt{\dfrac{2}{3}} = \dfrac{\sqrt{2}}{3e^t}.$

(*b*) $\|\mathbf{x}(t)\|^2 = \sin^2 t \cos^2 t + \cos^4 t + \sin^2 t = \cos^2 t(\sin^2 t + \cos^2 t) + \sin^2 t = \cos^2 t + \sin^2 t = 1.$ Hence,

$$\begin{aligned} \frac{d}{dt}\big(\mathbf{x}(t) \cdot \mathbf{x}(t)\big) &= 0 \\ \mathbf{x}(t) \cdot \mathbf{x}'(t) + \mathbf{x}'(t) \cdot (t) &= 0 \\ 2\mathbf{x}(t) \cdot \mathbf{x}'(t) &= 0 \\ \mathbf{x}(t) \cdot \mathbf{x}'(t) &= 0, \end{aligned}$$

i.e. $\mathbf{x}'(t)$ is orthogonal to $\mathbf{x}(t)$.

□

Problem 6. (20 points)

Let $\mathbf{x} : \mathbb{R} \to \mathbb{R}^3$ be a smooth curve with nonzero curvature. Derive the formula for the torsion

$$\tau = \frac{\begin{vmatrix} x_1'(t) & x_2'(t) & x_3'(t) \\ x_1''(t) & x_2''(t) & x_3''(t) \\ x_1'''(t) & x_2'''(t) & x_3'''(t) \end{vmatrix}}{\|\mathbf{x}'(t) \times \mathbf{x}''(t)\|^2}.$$

You may use without proof that

$$\mathbf{x}''(t) = \left(\frac{d}{dt} \|\mathbf{x}'(t)\| \right) \mathbf{T} + \kappa \|\mathbf{x}'(t)\|^2 \mathbf{N}$$

and

$$\kappa = \frac{\|\mathbf{x}'(t) \times \mathbf{x}''(t)\|}{\|\mathbf{x}'(t)\|^3}.$$

Solution. By definition of τ,

$$\tau = -\frac{d\mathbf{B}}{ds} \cdot \mathbf{N} = -\frac{d}{ds}(\mathbf{T} \times \mathbf{N}) \cdot \mathbf{N} = -\left(\frac{d\mathbf{T}}{ds} \times \mathbf{N} \right) \cdot \mathbf{N} - \left(\mathbf{T} \times \frac{d\mathbf{N}}{ds} \right) \cdot \mathbf{N} = \left(\frac{d\mathbf{N}}{ds} \times \mathbf{T} \right) \cdot \mathbf{N}$$

since $\dfrac{d\mathbf{T}}{ds} \times \mathbf{N}$ is orthogonal to $\mathbf{N}$ and hence the dot product is 0.

To proceed, we need to find $\dfrac{\mathbf{dN}}{ds}$.

$$
\begin{aligned}
\mathbf{x}''(t) &= \left(\frac{d}{dt}\|\mathbf{x}'(t)\|\right)\mathbf{T} + \kappa\|\mathbf{x}'(t)\|^2\mathbf{N} \\
\mathbf{x}'''(t) &= \left(\frac{d^2}{dt^2}\|\mathbf{x}'(t)\|\right)\mathbf{T} + \left(\frac{d}{dt}\|\mathbf{x}'(t)\|\right)\frac{\mathbf{dT}}{dt} + \frac{d\kappa\|\mathbf{x}'(t)\|^2}{dt}\mathbf{N} + \kappa\|\mathbf{x}'(t)\|^2\frac{d\mathbf{N}}{dt} \\
\frac{d\mathbf{N}}{dt} &= \frac{1}{\kappa\|\mathbf{x}'(t)\|^2}\left[\mathbf{x}'''(t) - \left(\frac{d^2}{dt^2}\|\mathbf{x}'(t)\|\right)\mathbf{T} - \left(\frac{d}{dt}\|\mathbf{x}'(t)\|\right)\frac{\mathbf{dT}}{dt} - \frac{d\kappa\|\mathbf{x}'(t)\|^2}{dt}\mathbf{N}\right] \\
\frac{d\mathbf{N}}{ds}\frac{ds}{dt} &= \frac{1}{\kappa\|\mathbf{x}'(t)\|^2}\left[\mathbf{x}'''(t) - \left(\frac{d^2}{dt^2}\|\mathbf{x}'(t)\|\right)\mathbf{T} - \left(\frac{d}{dt}\|\mathbf{x}'(t)\|\right)\frac{\mathbf{dT}}{dt} - \frac{d\kappa\|\mathbf{x}'(t)\|^2}{dt}\mathbf{N}\right] \\
\frac{d\mathbf{N}}{ds} &= \frac{1}{\kappa\|\mathbf{x}'(t)\|^3}\left[\mathbf{x}'''(t) - \left(\frac{d^2}{dt^2}\|\mathbf{x}'(t)\|\right)\mathbf{T} - \left(\frac{d}{dt}\|\mathbf{x}'(t)\|\right)\frac{\mathbf{dT}}{dt} - \frac{d\kappa\|\mathbf{x}'(t)\|^2}{dt}\mathbf{N}\right].
\end{aligned}
$$

Hence,

$$
\frac{d\mathbf{N}}{ds} \times \mathbf{T} = \frac{1}{\kappa\|\mathbf{x}'(t)\|^3}\left[\mathbf{x}'''(t) \times \mathbf{T} - \left(\frac{d}{dt}\|\mathbf{x}'(t)\|\right)\frac{\mathbf{dT}}{dt} \times \mathbf{T} - \frac{d\kappa\|\mathbf{x}'(t)\|^2}{dt}\mathbf{N} \times \mathbf{T}\right]
$$

since $\mathbf{T} \times \mathbf{T} = \mathbf{0}$. Also, since $\frac{d\mathbf{T}}{dt} = \left\|\frac{d\mathbf{T}}{dt}\right\|\mathbf{N}$, we have

$$
\left(\frac{d\mathbf{T}}{dt} \times \mathbf{T}\right) \cdot \mathbf{N} = 0 \text{ and } (\mathbf{N} \times \mathbf{T}) \cdot \mathbf{N} = 0.
$$

As a result,

$$
\begin{aligned}
\tau = \left(\frac{d\mathbf{N}}{ds} \times \mathbf{T}\right) \cdot \mathbf{N} &= \frac{1}{\kappa\|\mathbf{x}'(t)\|^3}\left(\mathbf{x}'''(t) \times \mathbf{T}\right) \cdot \mathbf{N} \\
&= \frac{1}{\kappa\|\mathbf{x}'(t)\|^4}\left(\mathbf{x}'''(t) \times \mathbf{x}'(t)\right) \cdot \frac{1}{\kappa}\frac{d\mathbf{T}}{ds} \\
&= \frac{1}{\kappa^2\|\mathbf{x}'(t)\|^4}\left(\mathbf{x}'''(t) \times \mathbf{x}'(t)\right) \cdot \frac{d\mathbf{T}}{dt}\frac{dt}{ds} \\
&= \frac{1}{\kappa^2\|\mathbf{x}'(t)\|^5}\left(\mathbf{x}'''(t) \times \mathbf{x}'(t)\right) \cdot \frac{d}{dt}\frac{\mathbf{x}'(t)}{\|\mathbf{x}'(t)\|} \\
&= \frac{1}{\kappa^2\|\mathbf{x}'(t)\|^5}\left(\mathbf{x}'''(t) \times \mathbf{x}'(t)\right) \cdot \left[\left(\frac{d}{dt}\frac{1}{\|\mathbf{x}'(t)\|}\right)\mathbf{x}'(t) + \frac{\mathbf{x}''(t)}{\|\mathbf{x}'(t)\|}\right] \\
&= \frac{1}{\kappa^2\|\mathbf{x}'(t)\|^6}\left(\mathbf{x}'''(t) \times \mathbf{x}'(t)\right) \cdot \mathbf{x}''(t) \\
&= \frac{\left(\mathbf{x}'''(t) \times \mathbf{x}'(t)\right) \cdot \mathbf{x}''(t)}{\|\mathbf{x}'(t) \times \mathbf{x}''(t)\|^2} \\
&= \frac{\begin{vmatrix} x'''(t) & y'''(t) & z'''(t) \\ x'(t) & y'(t) & z'(t) \\ x''(t) & y''(t) & z''(t) \end{vmatrix}}{\|\mathbf{x}'(t) \times \mathbf{x}''(t)\|^2} \\
&= \frac{\begin{vmatrix} x'(t) & y'(t) & z'(t) \\ x''(t) & y''(t) & z''(t) \\ x'''(t) & y'''(t) & z'''(t) \end{vmatrix}}{\|\mathbf{x}'(t) \times \mathbf{x}''(t)\|^2}
\end{aligned}
$$

□