MATH 2010E ADVANCED CALCULUS I
LECTURE 9

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14.3 — Continue on differentiability

Recall the definition of differentiability.

Definition 1. Let $D \subseteq \mathbb{R}^n$ be an open domain. Let $f : D \to \mathbb{R}$ be a real-valued function. Let $\mathbf{a} = (a_1, \ldots, a_n) \in D$. Then f is **differentiable at $\mathbf{x} = \mathbf{a}$** if

$$\lim_{\mathbf{h} \to \mathbf{0}} \frac{f(\mathbf{a} + \mathbf{h}) - f(\mathbf{a}) - \nabla f(\mathbf{a}) \cdot \mathbf{h}}{\|\mathbf{h}\|} = 0,$$

where $\nabla f(\mathbf{a}) = \big(f_{x_1}(\mathbf{a}), f_{x_2}(\mathbf{a}), \ldots, f_{x_n}(\mathbf{a})\big)$, called the **gradient vector** of f at $\mathbf{x} = \mathbf{a}$. If f is differentiable at every point in D, then f is **differentiable**.

Also, recall that the equation of the **tangent plane** of $x_{n+1} = f(\mathbf{x})$ at the point $\mathbf{x} = \mathbf{a}$ is

$$x_{n+1} = f(\mathbf{a}) + \nabla f(\mathbf{a}) \cdot (\mathbf{x} - \mathbf{a}).$$

Remark: The tangent plane

$$-f_{x_1}(\mathbf{a})x_1 - \cdots - f_{x_n}(\mathbf{a})x_n + x_{n+1} = f(\mathbf{a}) - f_{x_1}(\mathbf{a})a_1 - \cdots - f_{x_n}(\mathbf{a})a_n$$

has a normal vector $\big(-f_{x_1}(\mathbf{a}), \ldots, -f_{x_n}(\mathbf{a}), 1\big) = \big(-\nabla f(\mathbf{a}), 1\big)$.

Example 2. Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be such that $f(x, y) = x^2 + y^2$. Prove that f is differentiable, and find the equation of the tangent plane at $(x, y) = (1, 0)$.

Solution. At every $(a, b) \in \mathbb{R}^2$,

$$\begin{aligned}
&\lim_{(h,k)\to(0,0)} \frac{(a+h)^2 + (b+k)^2 - (a^2 + b^2) - (2a, 2b) \cdot (h, k)}{\sqrt{h^2 + k^2}} \\
&= \lim_{(h,k)\to(0,0)} \frac{h^2 + k^2}{\sqrt{h^2 + k^2}} = \lim_{(h,k)\to(0,0)} \sqrt{h^2 + k^2} = 0.
\end{aligned}$$

Therefore, f is differentiable.

At $(x, y) = (1, 0)$, the equation of the tangent plane is

$$\begin{aligned}
z &= (1^2 + 0^2) + \big(2(1), 2(0)\big) \cdot (x - 1, y - 0) \\
z &= 1 + 2(x - 1) \\
2x - z &= 1.
\end{aligned}$$

$\square$

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Note that Definition 1 depends on the definition and the existence of partial derivatives. Hence, it is desirable to have another definition of differentiability without appealing to partial derivatives.

Definition 3. Let $D \subseteq \mathbb{R}^n$ be an open domain. Let $f : D \to \mathbb{R}$ be a real-valued function. Let $\mathbf{a} = (a_1, \ldots, a_n) \in D$. Then f is **differentiable at** $\mathbf{x} = \mathbf{a}$ if there exists a vector $\mathbf{v} \in \mathbb{R}^n$ such that

$$\lim_{\mathbf{h}\to\mathbf{0}} \frac{f(\mathbf{a}+\mathbf{h}) - f(\mathbf{a}) - \mathbf{v}\cdot\mathbf{h}}{\|\mathbf{h}\|} = 0.$$

If f is differentiable at every point in D, then f is **differentiable**.

Using Definition 3, we can prove the following theorem.

Theorem 4. *If f is differentiable at $\mathbf{x} = \mathbf{a}$ in the sense of Definition 3, then*
(a) $\mathbf{f}_{x_i}(\mathbf{a})$ exists for all $i = 1, 2, \ldots, n$.
(b) $\mathbf{v} = \nabla f(\mathbf{a})$.

Proof. The existence of the limit

$$\lim_{\mathbf{h}\to\mathbf{0}} \frac{f(\mathbf{a}+\mathbf{h}) - f(\mathbf{a}) - \mathbf{v}\cdot\mathbf{h}}{\|\mathbf{h}\|} = 0$$

implies that no matter which path we take for $\mathbf{h} \to \mathbf{0}$, the limit is also 0.

Let $\mathbf{e}_i = (0, \ldots, 0, 1, 0, \ldots, 0)$ be the vector with all entries 0 except a 1 in the i-th entry. If we take the path $\mathbf{h} = t\mathbf{e}_i$, then

$$\begin{aligned}
\lim_{\substack{h=t\mathbf{e}_i\\ t\to 0}} \frac{f(\mathbf{a}+\mathbf{h}) - f(\mathbf{a}) - \mathbf{v}\cdot\mathbf{h}}{\|\mathbf{h}\|} = \lim_{t\to 0} \frac{f(\mathbf{a}+t\mathbf{e}_i) - f(\mathbf{a}) - tv_i}{|t|} &= 0\\
\lim_{t\to 0} \frac{f(\mathbf{a}+t\mathbf{e}_i) - f(\mathbf{a}) - tv_i}{t} &= 0\\
\lim_{t\to 0} \frac{f(\mathbf{a}+t\mathbf{e}_i) - f(\mathbf{a})}{t} - v_i &= 0\\
\lim_{t\to 0} \frac{f(\mathbf{a}+t\mathbf{e}_i) - f(\mathbf{a})}{t} &= v_i.
\end{aligned}$$

Since the existence of $\mathbf{v} = (v_1, \ldots, v_n)$ is guaranteed in Definition 3, the partial derivatives $f_{x_i}(\mathbf{a}) = v_i$ will also exist. Also, $\mathbf{v} = \big(f_{x_1}(\mathbf{a}), \ldots, f_{x_n}(\mathbf{a})\big) = \nabla f(\mathbf{a})$.

□

Theorem 5. *If f is differentiable at $\mathbf{a}$, then f is continuous at $\mathbf{a}$.*

Proof.

$$\begin{aligned}
\lim_{\mathbf{h}\to\mathbf{0}} \frac{f(\mathbf{a}+\mathbf{h}) - f(\mathbf{a}) - \nabla f(\mathbf{a})\cdot\mathbf{h}}{\|\mathbf{h}\|} &= 0\\
\lim_{\mathbf{h}\to\mathbf{0}} \big[f(\mathbf{a}+\mathbf{h}) - f(\mathbf{a}) - \nabla f(\mathbf{a})\cdot\mathbf{h}\big] &= 0\\
\lim_{\mathbf{h}\to\mathbf{0}} \big[f(\mathbf{a}+\mathbf{h}) - f(\mathbf{a})\big] - \nabla f(\mathbf{a})\cdot \lim_{\mathbf{h}\to\mathbf{0}} \mathbf{h} &= 0\\
\lim_{\mathbf{h}\to\mathbf{0}} \big[f(\mathbf{a}+\mathbf{h}) - f(\mathbf{a})\big] &= 0\\
\lim_{\mathbf{h}\to\mathbf{0}} f(\mathbf{a}+\mathbf{h}) &= f(\mathbf{a}),
\end{aligned}$$

which is precisely the definition that f is continuous at $\mathbf{a}$.

□

Conclusion:

$$\begin{array}{ccc} & f_{x_1},\ldots,f_{x_n} \text{ are continuous at } \mathbf{a} \ (f \text{ is } C^1 \text{ at } \mathbf{a}) & \\ & \Downarrow & \\ & f \text{ is differentiable at } \mathbf{a} & \\ & \Downarrow \qquad\qquad \Downarrow & \\ f \text{ is continuous at } \mathbf{a} & \not\Leftrightarrow & f_{x_1},\ldots,f_{x_n} \text{ exist at } \mathbf{a} \end{array}$$

Example 6. Determine whether $f(x,y) = \begin{cases} \dfrac{xy}{\sqrt{x^2+y^2}} & \text{if } (x,y) \neq 0 \\ 0 & \text{otherwise} \end{cases}$ is

(a) C^1 at $(0,0)$.
(b) differentiable at $(0,0)$.

Solution.

$$f_x(0,0) = \lim_{h\to 0} \frac{f(h,0)-f(0,0)}{h} = \lim_{h\to 0} \frac{0-0}{h} = 0.$$
$$f_y(0,0) = \lim_{h\to 0} \frac{f(0,h)-f(0,0)}{h} = \lim_{h\to 0} \frac{0-0}{h} = 0.$$

$$\begin{aligned} &\lim_{(h,k)\to(0,0)} \frac{f(h,k)-f(0,0)-\big(f_x(0,0),f_y(0,0)\big)\cdot(h,k)}{\sqrt{h^2+k^2}} \\ =& \lim_{(h,k)\to(0,0)} \frac{\dfrac{hk}{\sqrt{h^2+k^2}} - 0 - (0,0)\cdot(h,k)}{\sqrt{h^2+k^2}} \\ =& \lim_{(h,k)\to(0,0)} \frac{hk}{h^2+k^2} \end{aligned}$$

which does not exist since

$$\lim_{\substack{(h,k)=t(1,0)\\ t\to 0}} \frac{hk}{h^2+k^2} = \lim_{t\to 0} \frac{t\cdot 0}{t^2+0^2} = 0$$

and

$$\lim_{\substack{(h,k)=t(1,1)\\ t\to 0}} \frac{hk}{h^2+k^2} = \lim_{t\to 0} \frac{t\cdot t}{t^2+t^2} = \lim_{t\to 0} \frac{t^2}{2t^2} = \frac{1}{2}.$$

Therefore, f is not differentiable at $(0,0)$, and by the above conclusion, f is not C^1 at $(0,0)$.

□

Example 7. Determine whether $f(x,y) = \begin{cases} \dfrac{x^2y}{\sqrt{x^2+y^2}} & \text{if } (x,y) \neq 0 \\ 0 & \text{otherwise} \end{cases}$ is

(a) C^1 at $(0,0)$.
(b) differentiable at $(0,0)$.

Solution. When $(x,y) \neq (0,0)$,

$$f_x(x,y) = \frac{2xy}{\sqrt{x^2+y^2}} - \frac{2x^3y}{2\big(\sqrt{x^2+y^2}\big)^3} = \frac{x^3y+2xy^3}{\big(\sqrt{x^2+y^2}\big)^3},$$

and

$$f_y(x,y) = \frac{x^2}{\sqrt{x^2+y^2}} - \frac{2x^2y^2}{2\big(\sqrt{x^2+y^2}\big)^3} = \frac{x^4}{\big(\sqrt{x^2+y^2}\big)^3}.$$

When $(x, y) = (0, 0)$,

$$f_x(0,0) = \lim_{h\to 0} \frac{f(h,0) - f(0,0)}{h} = \lim_{h\to 0} \frac{0-0}{h} = 0,$$

and

$$f_y(0,0) = \lim_{h\to 0} \frac{f(0,h) - f(0,0)}{h} = \lim_{h\to 0} \frac{0-0}{h} = 0.$$

Note that $|x| = \sqrt{x^2} \leq \sqrt{x^2+y^2}$ and $|y| = \sqrt{y^2} \leq \sqrt{x^2+y^2}$, so

$$\begin{aligned}
|f_x(x,y)| &= \frac{|x^3y + 2xy^3|}{\left(\sqrt{x^2+y^2}\right)^3} \\
&\leq \frac{|x|^3|y| + 2|x||y|^3}{\left(\sqrt{x^2+y^2}\right)^3} \quad \text{(by triangle inequality)} \\
&\leq \frac{\left(\sqrt{x^2+y^2}\right)^3\sqrt{x^2+y^2} + 2\sqrt{x^2+y^2}\left(\sqrt{x^2+y^2}\right)^3}{\left(\sqrt{x^2+y^2}\right)^3} \\
&= 3\sqrt{x^2+y^2},
\end{aligned}$$

and

$$|f_y(x,y)| = \frac{|x|^4}{\left(\sqrt{x^2+y^2}\right)^3} \leq \frac{\left(\sqrt{x^2+y^2}\right)^4}{\left(\sqrt{x^2+y^2}\right)^3} = \sqrt{x^2+y^2}.$$

Since $\lim\limits_{(x,y)\to(0,0)} \sqrt{x^2+y^2} = 0$, by sandwich theorem,

$$\lim_{(x,y)\to(0,0)} f_x(x,y) = \lim_{(x,y)\to(0,0)} f_y(x,y) = 0.$$

Therefore, f_x and f_y are continuous at $(0, 0)$, i.e. f is C^1 at $(0, 0)$, and by the above conclusion, f is differentiable at $(0, 0)$.

□

Definition 8. Let $D \subseteq \mathbb{R}^n$ be an open domain. Let $f : D \to \mathbb{R}^m$ be a vector-valued function, i.e. $f = (f_1, f_2, \ldots, f_m)$ for some real-valued functions $f_i : D \to \mathbb{R}$. Let $\mathbf{a} = (a_1, \ldots, a_n) \in D$. Then f **is differentiable at $\mathbf{x} = \mathbf{a}$** if

$$\lim_{\mathbf{h}\to\mathbf{0}} \frac{f(\mathbf{a}+\mathbf{h}) - f(\mathbf{a}) - Df(\mathbf{a})\cdot\mathbf{h}}{\|\mathbf{h}\|} = \mathbf{0},$$

where $Df(\mathbf{a}) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1}(\mathbf{a}) & \frac{\partial f_1}{\partial x_2}(\mathbf{a}) & \cdots & \frac{\partial f_1}{\partial x_n}(\mathbf{a}) \\ \frac{\partial f_2}{\partial x_1}(\mathbf{a}) & \frac{\partial f_2}{\partial x_2}(\mathbf{a}) & \cdots & \frac{\partial f_2}{\partial x_n}(\mathbf{a}) \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1}(\mathbf{a}) & \frac{\partial f_m}{\partial x_2}(\mathbf{a}) & \cdots & \frac{\partial f_m}{\partial x_n}(\mathbf{a}) \end{pmatrix}$, called the **partial derivative matrix** of f at $\mathbf{x} = \mathbf{a}$, and $\mathbf{h}$ is taken as a column vector in the matrix multiplication $Df(\mathbf{a}) \cdot \mathbf{h}$. If f is differentiable at every point in D, then f is **differentiable**.

Remark: This definition follows from the fact that f is differentiable if and only if $f_1, f_2 \ldots, f_m$ are differentiable, and the limit in Definition 8 is precisely combining all the limits needed for defining the differentiability of $f_1, f_2, \ldots, f_m$.

14.4 — Chain rule

Theorem 9 (Chain rule)**.** *Let $D \subseteq \mathbb{R}^n$ be an open domain. Let $f : D \to \mathbb{R}^m$ be a vector-valued function. Let $\mathbf{a} \in D$. Let $E \subseteq \mathbb{R}^m$ be an open domain such that $f(\mathbf{a}) \in E$. Let $g : E \to \mathbb{R}^k$ be a vector valued function. If f is differentiable at $\mathbf{a}$, and g is differentiable at $f(\mathbf{a})$, then $g \circ f$ is differentiable at $\mathbf{a}$, and*

$$D(g \circ f)(\mathbf{a}) = Dg(f(\mathbf{a})) \cdot Df(\mathbf{a}).$$

Restricted to some simple cases:

- $n = 1$, $m = 3$, $k = 1$, i.e. $\mathbb{R} \to \mathbb{R}^3 \to \mathbb{R}$:
  Let $f(t) = \big(x(t), y(t), z(t)\big)$, and let $g = g(x, y, z)$, then chain rule says

$$\frac{dg}{dt}(a) = \frac{\partial g}{\partial x}\big(x(a), y(a), z(a)\big)\frac{dx}{dt}(a) + \frac{\partial g}{\partial y}\big(x(a), y(a), z(a)\big)\frac{dy}{dt}(a) + \frac{\partial g}{\partial z}\big(x(a), y(a), z(a)\big)\frac{dz}{dt}(a),$$

  or when f and g are both differentiable functions, the chain rule says

$$\frac{dg}{dt} = \frac{\partial g}{\partial x}\frac{dx}{dt} + \frac{\partial g}{\partial y}\frac{dy}{dt} + \frac{\partial g}{\partial z}\frac{dz}{dt}.$$

- $n = 2$, $m = 3$, $k = 1$, i.e. $\mathbb{R}^2 \to \mathbb{R}^3 \to \mathbb{R}$:
  Let $f(r, s) = \big(x(r, s), y(r, s), z(r, s)\big)$, and let $g = g(x, y, z)$ so that both f and g are differentiable functions, then chain rule says

$$\frac{\partial g}{\partial r} = \frac{\partial g}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial g}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial g}{\partial z}\frac{\partial z}{\partial r},$$

$$\frac{\partial g}{\partial s} = \frac{\partial g}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial g}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial g}{\partial z}\frac{\partial z}{\partial s}.$$

- $n = 2$, $m = 1$, $k = 3$, i.e. $\mathbb{R}^2 \to \mathbb{R} \to \mathbb{R}^3$:
  Let $f = f(r, s)$, and let $g(f) = \big(g_1(f), g_2(f), g_3(f)\big)$ so that both f and g are differentiable functions, then chain rule says

$$\frac{\partial g_1}{\partial r} = \frac{dg_1}{df}\frac{\partial f}{\partial r}, \qquad \frac{\partial g_1}{\partial s} = \frac{dg_1}{df}\frac{\partial f}{\partial s},$$

$$\frac{\partial g_2}{\partial r} = \frac{dg_2}{df}\frac{\partial f}{\partial r}, \qquad \frac{\partial g_2}{\partial s} = \frac{dg_2}{df}\frac{\partial f}{\partial s},$$

$$\frac{\partial g_3}{\partial r} = \frac{dg_3}{df}\frac{\partial f}{\partial r}, \qquad \frac{\partial g_3}{\partial s} = \frac{dg_3}{df}\frac{\partial f}{\partial s}.$$

- $n = 2$, $m = 2$, $k = 3$, i.e. $\mathbb{R}^2 \to \mathbb{R}^2 \to \mathbb{R}^3$:
  Let $f = \big(x(r, s), y(r, s)\big)$, and let $g(f) = \big(g_1(x, y), g_2(x, y), g_3(x, y)\big)$ so that both f and g are differentiable functions, then chain rule says

$$\frac{\partial g_1}{\partial r} = \frac{\partial g_1}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial g_1}{\partial y}\frac{\partial y}{\partial r}, \qquad \frac{\partial g_1}{\partial s} = \frac{\partial g_1}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial g_1}{\partial y}\frac{\partial y}{\partial s},$$

$$\frac{\partial g_2}{\partial r} = \frac{\partial g_2}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial g_2}{\partial y}\frac{\partial y}{\partial r}, \qquad \frac{\partial g_2}{\partial s} = \frac{\partial g_2}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial g_2}{\partial y}\frac{\partial y}{\partial s},$$

$$\frac{\partial g_3}{\partial r} = \frac{\partial g_3}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial g_3}{\partial y}\frac{\partial y}{\partial r}, \qquad \frac{\partial g_3}{\partial s} = \frac{\partial g_3}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial g_3}{\partial y}\frac{\partial y}{\partial s}.$$

Example 10. Let $f(r,s) = \big(x(r,s), y(r,s), z(r,s)\big) = (3e^r \sin s, 3e^r \cos s, 4e^r)$, and let $g(x,y,z) = \sqrt{x^2+y^2+z^2}$. Find

(*a*) $\dfrac{\partial g}{\partial r}$ and $\dfrac{\partial g}{\partial s}$.

(*b*) $\dfrac{\partial g}{\partial r}(0,0)$ and $\dfrac{\partial g}{\partial s}(0,0)$.

Solution. (*a*) *Method* 1 (*direct substitution*).

$$g(r,s) = \sqrt{9e^{2r}\sin^2 s + 9e^{2r}\cos^2 s + 16e^{2r}} = 5e^r,$$

so

$$\frac{\partial g}{\partial r} = 5e^r \qquad \text{and} \qquad \frac{\partial g}{\partial s} = 0.$$

Method 2 (*chain rule*).

$$\frac{\partial g}{\partial x} = \frac{2x}{2\sqrt{x^2+y^2+z^2}} = \frac{3e^r \sin s}{\sqrt{9e^{2r}\sin^2 s + 9e^{2r}\cos^2 s + 16e^{2r}}} = \frac{3e^r \sin s}{5e^r} = \frac{3\sin s}{5},$$

$$\frac{\partial g}{\partial y} = \frac{2y}{2\sqrt{x^2+y^2+z^2}} = \frac{3e^r \cos s}{5e^r} = \frac{3\cos s}{5},$$

$$\frac{\partial g}{\partial z} = \frac{2z}{2\sqrt{x^2+y^2+z^2}} = \frac{4e^r}{5e^r} = \frac{4}{5}.$$

Therefore, by chain rule,

$$\frac{\partial g}{\partial r} = \frac{3\sin s}{5}(3e^r \sin s) + \frac{3\cos s}{5}(3e^r \cos s) + \frac{4}{5}(4e^r) = 5e^r,$$

$$\frac{\partial g}{\partial s} = \frac{3\sin s}{5}(3e^r \cos s) + \frac{3\cos s}{5}(-3e^r \sin s) + \frac{4}{5}(0) = 0.$$

(*b*) *Method* 1 (*use* (*a*)).

$$\frac{\partial g}{\partial r}(0,0) = 5e^r\big|_{(r,s)=(0,0)} = 5 \qquad \text{and} \qquad \frac{\partial g}{\partial s}(0,0) = 0\big|_{(r,s)=(0,0)} = 0.$$

Method 2 (*chain rule*). At $(r,s) = (0,0)$, $(x,y,z) = (3e^0 \sin 0, 3e^0 \cos 0, 4e^0) = (0,3,4)$.

$$\frac{\partial g}{\partial x}(0,3,4) = \left.\frac{2x}{2\sqrt{x^2+y^2+z^2}}\right|_{(x,y,z)=(0,3,4)} = 0,$$

$$\frac{\partial g}{\partial y}(0,3,4) = \left.\frac{2y}{2\sqrt{x^2+y^2+z^2}}\right|_{(x,y,z)=(0,3,4)} = \frac{3}{5},$$

$$\frac{\partial g}{\partial z}(0,3,4) = \left.\frac{2z}{2\sqrt{x^2+y^2+z^2}}\right|_{(x,y,z)=(0,3,4)} = \frac{4}{5}.$$

Therefore, by chain rule,

$$\begin{aligned}\frac{\partial g}{\partial r}(0,0) &= 0\left(3e^r \sin s|_{(r,s)=(0,0)}\right) + \frac{3}{5}\left(3e^r \cos s|_{(r,s)=(0,0)}\right) + \frac{4}{5}\left(4e^r|_{(r,s)=(0,0)}\right)\\ &= 0(0) + \frac{3}{5}(3) + \frac{4}{5}(4) = 5,\end{aligned}$$

$$\begin{aligned}\frac{\partial g}{\partial s}(0,0) &= 0\left(3e^r \cos s|_{(r,s)=(0,0)}\right) + \frac{3}{5}\left(-3e^r \sin s|_{(r,s)=(0,0)}\right) + \frac{4}{5}\left(0|_{(r,s)=(0,0)}\right)\\ &= 0(3) + \frac{3}{5}(0) + \frac{4}{5}(0) = 0.\end{aligned}$$

□

Example 11. Let $F(x,y,z)=0$ be a smooth level surface that implicitly defines z as a function of x and y. In other words, $z=f(x,y)$ and $F(x,y,f(x,y))=0$. Find $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$.

Solution. This can be viewed as the case $\mathbb{R}^2 \to \mathbb{R}^3 \to \mathbb{R}$, and by differentiating the equation $0=F(x,y,z)$ using chain rule, we have

$$0=\frac{\partial F}{\partial x}\frac{\partial x}{\partial x}+\frac{\partial F}{\partial y}\frac{\partial y}{\partial x}+\frac{\partial F}{\partial z}\frac{\partial z}{\partial x} \quad\Rightarrow\quad 0=F_x+F_z\frac{\partial z}{\partial x},$$

$$0=\frac{\partial F}{\partial x}\frac{\partial x}{\partial y}+\frac{\partial F}{\partial y}\frac{\partial y}{\partial y}+\frac{\partial F}{\partial z}\frac{\partial z}{\partial y} \quad\Rightarrow\quad 0=F_y+F_z\frac{\partial z}{\partial y}.$$

Therefore,

$$\frac{\partial z}{\partial x}=-\frac{F_x}{F_z} \qquad\text{and}\qquad \frac{\partial z}{\partial y}=-\frac{F_y}{F_z}$$

if $F_z\neq 0$.

$\square$

Theorem 12 (Implicit function theorem)**.** *Let $D\subseteq\mathbb{R}^3$ be an open domain. Let $F:D\to\mathbb{R}$ be a real-valued function. Let (a,b,c) be a point on the level surface $F(x,y,z)=k$. Then this level surface implicitly defines z as a function of x and y locally near $(x,y,z)=(a,b,c)$ if*

- *F is a C^1 function, i.e. f_x, f_y, f_z exist and are continuous in D, and*
- *$F_z(a,b,c)\neq 0$.*

Example 13. In polar coordinates in $\mathbb{R}^3$, we have

$$(x,y,z)=(r\cos\theta\cos\phi, r\sin\theta\cos\phi, r\sin\phi).$$

If $f=f(x,y,z)$ is a smooth function, find $\dfrac{\partial f}{\partial r},\dfrac{\partial f}{\partial \theta},\dfrac{\partial f}{\partial \phi}$.

Solution.

$$\frac{\partial f}{\partial r}=(\cos\theta\cos\phi)f_x+(\sin\theta\cos\phi)f_y+(\sin\phi)f_z.$$

$$\frac{\partial f}{\partial \theta}=(-r\sin\theta\cos\phi)f_x+(r\cos\theta\cos\phi)f_y.$$

$$\frac{\partial f}{\partial \phi}=(-r\cos\theta\sin\phi)f_x+(-r\sin\theta\sin\phi)f_y+(r\cos\phi)f_z.$$

$\square$